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Holomorphic Anomaly Of Unitarity Cuts And One-Loop Gauge Theory Amplitudes

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We show how the holomorphic anomaly found in hep-th/0409245 can be used to efficiently compute certain classes of unitarity cuts of one-loop $\mathcal{N} = 4$ amplitudes of gluons. These classes include all cuts of n-gluon one-loop MHV amplitudes and of n-gluon next-to-MHV amplitudes with helicities $(1^+, 2^+, 3^+, 4^-, \ldots, n^-)$. As an application of this method, we present the explicit computation of the $(1, 2, 3)$-cut of the n-gluon one-loop $\mathcal{N} = 4$ leading-color amplitude $A_{n;1}(1^+, 2^+, 3^+, 4^-, \ldots, n^-)$. The answer is given in terms of scalar box functions and provides information about the corresponding amplitudes. A possible way to generalize this method to all kinds of unitarity cuts is also discussed.

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1. Introduction

The perturbative analysis of gauge theories has been a very important tool to compare theories and experiments. A great among of effort has been put on the calculation of scattering amplitudes in QCD, where the perturbative analysis is useful in the high-energy regime. Although QCD is well tested in this regime, calculating these amplitudes is important in order to subtract the QCD background from possible new physics at colliders.

At tree-level and one-loop level, new techniques have made possible calculations that are practically impossible by standard textbook approaches (for a review see for example [1]).

At one-loop level, one such technique is based on the "supersymmetric" decomposition of QCD amplitudes of gluons. Namely, a one-loop amplitude in $\mathcal{N} = 4$ super Yang-Mills contains the QCD amplitude plus contributions from fermions and scalars running in the loop. Combining all fermions with some scalars, $\mathcal{N} = 1$ chiral multiplets can be formed. This leaves only the contribution of scalars running in the loop. Therefore, once the supersymmetric amplitudes are known, the complicated QCD calculation is reduced to that of a scalar running in the loop, which is much simpler. Thus, calculations of supersymmetric amplitudes of gluons is also a subject of phenomenological interest.

One-loop amplitudes of gluons in $\mathcal{N} = 4$ gauge theory satisfy three remarkable properties. The first one is that all integrals that can appear in a direct Feynman graph calculation can be reduced to a set of known integrals in dimensional regularization [2,3,4]. These are known as scalar box integrals [5,6].

The second property comes from the study of the analytic structure of scalar box integrals. These integrals are multi-valued functions, i.e., they have branch cuts in the space of kinematical invariants. Moreover, there is no linear combination of these functions, with rational coefficients in the kinematical invariants, which is single-valued, i.e., a rational function [7,8].

This gives the second property: All $\mathcal{N} = 4$ amplitudes can be determined completely once their branch cuts and monodromies are known. Amplitudes with this property are said to be four-dimensional cut-constructible [7,8].

The third property, also shared by tree-level amplitudes and quite possibly by higher loops, is that once the amplitudes are transformed to twistor space, they turn out to be localized on simple algebraic sets [9,10,11,12,13]. The algebraic sets can be described as just

unions of "lines", i.e., $\mathbb{CP}^1$'s linearly embedded in twistor space, $\mathbb{CP}^3$. At tree-level, all amplitudes were constructed from a string theory with twistor space as its target space[1]. This led to a new prescription for calculating *all* tree amplitudes in terms of maximal helicity violating or MHV amplitudes continued off-shell and connected by Feynman propagators [11]; these are called MHV diagrams.

At one-loop, the simplest twistor picture, originally proposed in [9], implies that MHV amplitudes should be localized on two lines in twistor space [9]. Each line supports a tree-level MHV amplitude. A straightforward generalization of the tree-level construction of [11] suggests that the two lines or off-shell MHV amplitudes should be connected by two Feynman propagators to make up the loop amplitude. This idea was explicitly carried out in [14] and found to correctly reproduce the known n-gluon one-loop MHV amplitudes, first computed in [7].

On the other hand, the twistor space support of one-loop MHV amplitudes was studied in [12] by considering the differential equations they obey. The result involved configurations were all gluons except one were localized on two lines. This puzzle was resolved in [13]; a holomorphic anomaly affects the result of the differential operators acting on one-loop amplitudes. The way this anomaly works was most transparent on the unitarity cuts of one-loop amplitudes [13].

It is the aim of this paper to exploit the holomorphic anomaly to compute certain unitarity cuts of one-loop $\mathcal{N}=4$ amplitudes. The importance of this is that once the cuts are know explicitly so is the amplitude due to the cut-constructibility property.

The basic fact we observe is that any differential operator designed to annihilate the unitarity cut but that fails to do so due to the holomorphic anomaly, it is guaranteed to produce a rational function.

On the other hand, any unitarity cut can be written as the imaginary part of the amplitude in some suitable kinematical regime. This implies that it is given by the imaginary part of some combination of scalar box integrals with rational coefficients.

We then show that any differential operator that produces a rational function on the cut via the holomorphic anomaly annihilates the coefficients of the scalar box integrals in the amplitude[2]. This implies that the operator only acts on the monodromies of the

[1] There is growing evidence that higher-loop amplitudes might as well be computed by some sort of string theory in twistor space.

[2] To be precise, this statement is true only for scalar box *functions*, which are scalar box integrals nicely normalized.

scalar box integrals. These are just logarithms of rational functions[3]. Therefore, the result of applying the operator to the imaginary part is a rational function with some unknown coefficients.

We give a simple way of comparing the two formulas and therefore of extracting the unknown coefficients unambiguously. This is essentially a generalization of the proof of cut-constructibility of the amplitudes.

Using this method we find the explicit form of the $(1,2,3)$ cut of the one-loop leading-color partial amplitude $A_{n:1}(1^+,2^+,3^+,4^-,\ldots,n^-)$ in terms of only four scalar box functions.

This paper is organized as follows. In section 2, we briefly review one-loop amplitudes in $\mathcal{N}=4$ gauge theories and their cuts. Emphasis is made in the way they can be written in terms of scalar box functions. In section 2.1, the holomorphic anomaly is explained in the context of the unitarity cuts. Collinear operators are introduced and it is explained how they localize the cut integral to give a rational function. In section 2.2, we give the recipe for extracting information from this rational function by comparing it to the action of the collinear operator on the scalar box functions. The most general classes of cuts in which this method is directly applicable is also given. These involve all possible cuts in one-loop MHV amplitudes and in one-loop next-to-MHV amplitudes with helicities $(1^+,2^+,3^+,4^-,\ldots,n^-)$.

In section 3, the computation of the $(1,2,3)$ cut of $A_{n:1}(1^+,2^+,3^+,4^-,\ldots,n^-)$ is presented following the general method explained in section 2. In section 4, we make some consistency checks on the result of section 3. In section 5, we discuss our results and speculate on the way this can be generalized to all cuts and thus provide a way of computing *all* one-loop amplitudes. In appendix A, a detailed description of scalar box functions and their imaginary parts is given. Finally, appendix B contains some technical details of the computation in section 3.

2. General Framework

One-loop amplitudes of gluons in $\mathcal{N}=4$ $U(N)$ gauge theories depend on the momentum (p), helicity (h), and color index (a) of each of the external gluons. Consider

[3] The four-mass scalar box integral is an exception to this and it is treated separately.

the n-gluon amplitude $A_n(\{p_i, h_i, a_i\})$. It is very useful to separate the color structure explicitly as follows

$$A_n(\{p_i, h_i, a_i\}) = \sum_{c=1}^{[n/2]+1} \sum_{\sigma} Gr_{n;c}(\sigma) A_{n;c}(\sigma) \tag{2.1}$$

where

$$Gr_{n;c}(\sigma) = \text{Tr}\ (T^{\sigma(a_1)} \dots T^{\sigma(a_{c-1})}) \text{Tr}\ (T^{\sigma(a_c)} \dots T^{\sigma(a_n)}) \tag{2.2}$$

and σ is the set of possible permutations of n-gluons mod out by the symmetries of $Gr_{n;c}(\sigma)$.

This decomposition is useful because the partial amplitudes $A_{n;c}(\sigma)$ do not have color structure and the Feynman graphs used for their computation are color-ordered. (For a nice review see for example [1].)

Here we study only the leading-color partial amplitudes[4] $A_{n;1}$ for which $Gr_{n:1}(1) = N\text{Tr}\ (T^{a_1} \dots T^{a_n})$. In what follows, we refer to $A_{n;1}$ simply as the n-gluon one-loop amplitude.

All n-gluon one-loop MHV amplitudes are known explicitly [7]. These are amplitudes where two gluons have negative (positive) helicity and $n-2$ have positive (negative) helicity; we refer to these as mostly plus (mostly minus) MHV amplitudes. Out of the non-MHV amplitudes, only the simplest one has been computed [8], i.e., the six-gluon amplitude with three plus and three minus helicity gluons.

Even though not much is known about general amplitudes, two important properties are known. One is that all possible Feynman integrals that can enter in a textbook calculation of these amplitudes can be expressed in terms of five families of functions that are explicitly known [7]. These are the scalar box functions (appendix A is devoted to a careful description of these functions as well as their relation to the scalar box integrals mentioned in the introduction):

$$\{F^{1m}_{n:i}, \quad F^{2m\ e}_{n:r;i}, \quad F^{2m\ h}_{n:r;i}, \quad F^{3m}_{n:r:r';i}, \quad F^{4m}_{n:r:r':r'';i}\}. \tag{2.3}$$

The range of the indices $\{r, r', r'', i\}$ depends on the symmetries of the functions and it is discussed in appendix A. What is important to us at this point is that all of them are known very explicitly.

[4] It turns out that subleading-color amplitudes are determined in terms of the leading-color amplitudes[7].

Any n-gluon one-loop amplitude can be written as a linear combination of the scalar box functions (2.3). More explicitly,

$$A_{n;1} = \sum_{i=1}^{n}\left(b_i F^{1m}_{n:i} + \sum_r c_{r,i} F^{2m\ e}_{n:r;i} + \sum_r d_{r,i} F^{2m\ h}_{n:r;i} + \sum_{r,r'} g_\alpha F^{3m}_{n:r:r';i} + \sum_{r,r',r''} f_\beta F^{4m}_{n:r:r':r'';i}\right) \tag{2.4}$$

where $\alpha = \{r, r', i\}$ and $\beta = \{r, r', r'', i\}$. The coefficients in this formula naturally have a factor of $-2ic_\Gamma$, where c_Γ is a constant that shows up from the dimensional regularization procedure of the amplitude. We choose to ignore all these overall factors as they can easily be introduced at the end.

The fact that $A_{n;1}$ can be written as in (2.4) implies that the task of computing the amplitude is thus reduced to that of computing the coefficients. These coefficients are rational functions of the kinematical invariants of the external gluons. All information about the helicity of the external gluons is encoded in these coefficients. For example, one-loop MHV amplitudes have all coefficients either zero or equal to the corresponding tree-level MHV amplitude [7].

These coefficients are expected to have the simplest form in the spinor-helicity formalism [15,16,17] which we know briefly review. Here we follow the conventions stated in section 2 of [9].

In four dimensions, the momentum vector p of a gluon can be written as a bispinor of the form $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_a$. Spinor inner products are denoted as $\langle \lambda, \lambda' \rangle = \epsilon_{ab}\lambda^a \lambda'^b$ and $[\widetilde{\lambda}, \widetilde{\lambda}'] = \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}^{\dot{a}}\widetilde{\lambda}'^{\dot{b}}$. In terms of these, the inner product of the momenta of two gluons, p and q is expressed by $2p \cdot q = \langle \lambda_p, \lambda_q \rangle [\widetilde{\lambda}_p, \widetilde{\lambda}_q]$. In order to avoid cluttering the equations, it is useful to write $\langle p\ q \rangle$ and $[p\ q]$ instead of $\langle \lambda_p, \lambda_q \rangle$ and $[\widetilde{\lambda}_p, \widetilde{\lambda}_q]$ respectively.

The main simplification arises because the polarization vector of a gluon with momentum $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_a$ can be written as: $\epsilon_{a\dot{a}} = \lambda_a \widetilde{\nu}_{\dot{a}} / [\widetilde{\lambda}, \widetilde{\nu}]$ for negative helicity and as $\epsilon_{a\dot{a}} = \nu_a \widetilde{\lambda}_{\dot{a}} / \langle \nu, \lambda \rangle$ for positive helicity, where ν is a fixed reference spinor. This reference spinor can be wisely chosen to produce simple formulas. See page 16 of [18] for an example of such a simplification in the five-gluon tree-level amplitude.

The final result of this is that the coefficients in (2.4) are rational functions of spinor products of the external gluons.

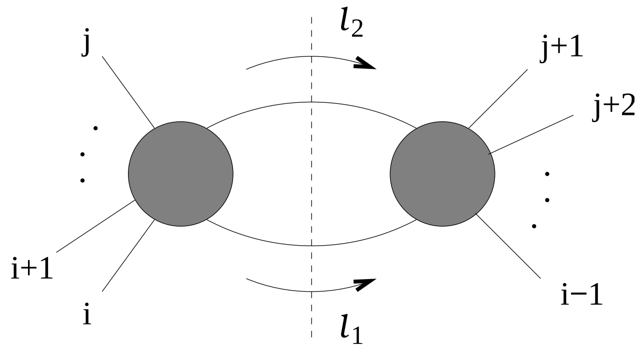

Fig. 1: Representation of the cut integral. Left and right tree-level amplitudes are on-shell. Internal lines represent the legs coming from the cut propagators.

The second remarkable property is that these amplitudes are four-dimensional cut-constructible. This means that their four-dimensional branch cuts and the corresponding monodromies determine them uniquely. This is why computing the unitarity cuts is a way of finding the amplitudes.

Let us turn to the computations of the cuts. Consider for example, the cut in the $(i, i+1, \ldots, j-1, j)$-channel. This is given by the "cut" integral

$$C_{i,i+1,\ldots,j-1,j} = \int d\mu A^{\text{tree}}((-\ell_1), i, i+1, \ldots, j-1, j, (-\ell_2))A^{\text{tree}}(\ell_2, j+1, j+2, \ldots, i-2, i-1, \ell_1) \tag{2.5}$$

where $d\mu$ is the Lorentz invariant phase space measure of two light-like vectors (ℓ_1, ℓ_2) constrained by momentum conservation. We find it useful to define ℓ_1 and ℓ_2 as in fig. 1. We think about the flow of energy as going from one side of the cut (dashed line in fig. 1) to the other. Helicity assignments are for incoming particles[5].

These cuts (2.5) compute the imaginary part of the amplitude in some suitable chosen kinematical regime. To describe this regime more explicitly we have to consider the possible kinematical invariants that the scalar box functions can depend on. Due to the color-order, the only invariants involve consecutive sets of gluons and are usually denoted by $t_i^{[r]} = (p_i + p_{i+1} + \ldots + p_{i+r-1})^2$. In other words, the invariants are characterized by chains of gluons labelled by the first gluon in the chain (i) and by the length of the chain (r).

[5] These conventions are slightly different from those used in [7] but, of course, the results are independent of the choice.

The cut $C_{i,i+1,\ldots,j-1,j}$ computes the imaginary part of (2.4) in the unphysical kinematical regime where $t_i^{[j-i+1]} = (p_i + p_{i+1} + \ldots + p_j)^2$ is positive and all other invariants are negative [7].

It is now clear that computing $C_{i,i+1,\ldots,j-1,j}$ provides information about the coefficients in (2.4) via

$$C_{i,i+1,\ldots,j-1,j} = \mathrm{Im}|_{t_i^{[j-i+1]}>0} A_{n;1}. \tag{2.6}$$

Indeed, this is the way that all known one-loop $\mathcal{N}=4$ amplitudes have been computed or checked [7,8].

We know turn to the problem of computing $C_{i,i+1,\ldots,j-1,j}$ using the holomorphic anomaly found in [13].

2.1. Twistor Space Support, Collinear Operators And Holomorphic Anomaly

In [9], a remarkable conjecture about the localization of $\mathcal{N}=4$ amplitudes in twistor space was made. Amplitudes of n gluons at L-loop order with q gluons of negative helicity were conjectured to be localized on curves of genus $g \leq L$ and degree $d = q + L - 1$ when transformed to twistor space.

This conjecture was further explored in [10,11,12,13]. In particular, it was shown in [11] that all tree-level amplitudes could be computed from configurations of unions of $q-1$ lines, i.e. $\mathbb{CP}^1$'s. Mostly plus MHV one-loop amplitudes were considered in [12]. They turn out to be localized on unions of lines after a holomorphic anomaly in the analysis is taken into account [13]. This is in perfect agreement with the original picture of [9].

In [9], a method for testing localization of gluons in twistor space was proposed. Suppose that gluons i,j, and k are collinear in twistor space[6]. In other words, the twistor transform of the amplitude vanishes unless these gluons lie on the same $\mathbb{CP}^1$ inside $\mathbb{CP}^3$. This means that if $Z^I = (\lambda^a, \mu^{\dot a})$, with $\mu_{\dot a} = -i\partial/\partial\widetilde{\lambda}^{\dot a}$, are coordinates in twistor space, then the vector $V_L = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K$ vanishes. In Minkowski space, this is a vector of differential operators; each component is supposed to annihilate any amplitude in which gluons i,j, and k are collinear in twistor space.

Let us choose the dotted components, $L = \dot a$, and construct the following spinor-valued operator

$$F_{ijk;\dot a} = \epsilon_{IJK\dot a} Z_i^I Z_j^J Z_k^K. \tag{2.7}$$

[6] This collinearity condition has nothing to do with the usual meaning of the word collinear in the scattering amplitude literature. There, two gluons are collinear if the corresponding momenta are proportional.

More explicitly,

$$F_{ijk;\dot a} = \langle i\ j\rangle \frac{\partial}{\partial \widetilde\lambda_k^{\dot a}} + \langle k\ i\rangle \frac{\partial}{\partial \widetilde\lambda_j^{\dot a}} + \langle j\ k\rangle \frac{\partial}{\partial \widetilde\lambda_i^{\dot a}}. \tag{2.8}$$

The advantage of this choice if that the operator (2.8) is of first order while for $L = a$ is of second order.

In the following, it will be convenient to introduce a fixed arbitrary negative chirality spinor $\eta^{\dot a}$ and consider

$$[F_{ijk}, \eta] = \epsilon^{\dot a\dot b}\eta_{\dot a}F_{ijk;\dot b}. \tag{2.9}$$

Note that the brackets in (2.9) are meant to indicate the inner product of two negative chirality spinors and not the commutator of operators.

A very important example is when $[F_{ijk}, \eta]$ acts on a function G whose dependence on the three gluons i, j, and k is only through $\{\lambda_i, \lambda_j, \lambda_k, (p_i + p_j + p_k)\}$. We want to show that $[F_{ijk}, \eta]G = 0$. Upon using the chain rule we find that $[F_{ijk}, \eta]G$ is proportional to the positive chirality spinor

$$\nu = \langle i\ j\rangle\lambda_k + \langle k\ i\rangle\lambda_j + \langle j\ k\rangle\lambda_i. \tag{2.10}$$

The idea is to show that ν is zero. This is equivalent to showing that $\langle \nu, \chi\rangle$ is zero for any χ. But this is exactly equal to Schouten's identity

$$\langle i\ j\rangle\langle k\ \chi\rangle + \langle k\ i\rangle\langle j\ \chi\rangle + \langle j\ k\rangle\langle i\ \chi\rangle = 0. \tag{2.11}$$

In this derivation we have assumed that the operator acts trivially on $\lambda_i, \lambda_j, \lambda_k$. However, as shown in [13], this might not be the case when G is a one-loop amplitude or its unitarity cut.

In [13], it was shown that at one-loop there are situations in which gluons i, j, and k are collinear in twistor space and yet the operator (2.7) does not annihilate the amplitude. This was most clearly explained by considering not the amplitude but its unitarity cuts (2.5).

Consider the cut (2.5). For simplicity, let us assume that the particles in the loop are gluons. This restriction is useful because both tree-level amplitudes in (2.5) only involve gluons and can be expanded in terms of the MHV diagrams of [11]. Moreover, in the main example of this paper discussed in the next section, only gluons can actually propagate in the loop. The case when scalars or fermions run in the loop is slightly more complicated, for it involves generalizations of MHV diagrams [19,20,21,22].

As explained in [13], the holomorphic anomaly only affects the action of collinear operators that involve gluons next to the internal ones (for example, gluons $i-1$, i, j, and $j+1$ in (2.5)). The reason is that the anomaly shows up only when the integrant of (2.5) has a pole when the momentum of an internal gluon becomes proportional that of an external gluon. Due to the color-order, this only happens for adjacent gluons.

The classes of cuts we are interested in this paper are those for which any of the special gluons is collinear (in twistor space) with two more external gluons.

More explicitly, cuts belonging to this class[7] are those for which one of the tree-level amplitudes in (2.5) is a mostly plus MHV amplitude. Recall that mostly plus tree-level MHV amplitudes are manifestly localized on lines [9]. Therefore, all gluons participating in the cut are collinear. Of course, at least three external gluons should participate in order to satisfy the criterion[8].

Let the left tree amplitude in (2.5) be the mostly plus MHV amplitude. Assume that gluons k and m have negative helicity,

$$\begin{aligned} &C_{i,i+1,\ldots,j-1,j} = \\ &\int d\mu \, A^{\rm treeMHV}_{km}((-\ell_1), i, (i+1), \ldots, j, (-\ell_2)) A^{\rm tree}(\ell_2, j+1, j+2, \ldots, i-1, \ell_1) \end{aligned} \tag{2.12}$$

where gluons $j+1$ through $i-1$ can have any helicity. Note that k and m can be any gluons, including ℓ_1 and ℓ_2.

Let us write the left tree amplitude explicitly [23],

$$A^{\rm treeMHV}_{km}((-\ell_1), i, (i+1), \ldots, j, (-\ell_2)) = \frac{\langle k\ m\rangle^4}{\langle \ell_1, i\rangle\langle i\ i+1\rangle \cdot \langle j-1\ j\rangle\langle j\ \ell-2\rangle}. \tag{2.13}$$

Using this in (2.5) we have

$$\begin{aligned} &C_{i,i+1,\ldots,j-1,j} = \\ &\int d\mu \frac{\langle k\ m\rangle^4}{\langle i\ i+1\rangle \ldots \langle j-1\ j\rangle} \frac{1}{\langle \ell_1\ i\rangle\langle j\ \ell_2\rangle} A^{\rm tree}(\ell_2, j+1, j+2, \ldots, i-1, \ell_1). \end{aligned} \tag{2.14}$$

Now consider the action of the collinear operator (2.9) for gluons i, $i+1$ and $i+2$, i.e.,

$$[F_{i,i+1,i+2}, \eta] = \langle i\ i+1\rangle[\partial_{i+2}, \eta] + \langle i+1\ i+2\rangle[\partial_i, \eta] + \langle i+2\ i\rangle[\partial_{i+1}, \eta] \tag{2.15}$$

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[7] See section 5 for possible generalizations.

[8] Two-particle cuts are special. They have singularities and make sense only in some regularization scheme. However, no scalar box function has cuts *only* in two-particle channels; therefore, studying all other channels must suffice to determine the whole amplitude.

where $\partial_k = \partial/\partial\widetilde{\lambda}_k$.

Naively, $[F_{i,i+1,i+2}, \eta]C_{i,i+1,\ldots,j-1,j} = 0$, but as pointed out in [13], the presence of the pole $1/\langle \ell_1 \; i\rangle$ makes the action of (2.15) nontrivial.

The basic idea is that the action of $[F_{i,i+1,i+2}, \eta]$ on the pole produces a delta function via[9]

$$[d\overline{\lambda}_{\ell_1}, \partial_{\ell_1}] \frac{1}{\langle \lambda_{\ell_1}, \lambda_i\rangle} = d\overline{\lambda}^{\dot{a}}_{\ell_1} \frac{\partial}{\partial \overline{\lambda}^{\dot{a}}_{\ell_1}} \frac{1}{\langle \lambda_{\ell_1}, \lambda_i\rangle} = 2\pi\overline{\delta}(\langle \lambda_{\ell_1}, \lambda_i\rangle), \tag{2.16}$$

where we have introduced a $(0,1)$-form $\overline{\delta}(z) = -id\overline{z}\,\delta^2(z)$.

The main simplification arises because the integral over the Lorentz invariant phase space is an integral over a two sphere and the delta function produced by $[F_{i,i+1,i+2}, \eta]$ is enough to localize the integral completely. In other words, evaluating the action of $[F_{i,i+1,i+2}, \eta]$ on $C_{i,i+1,\ldots,j-1,j}$ does not involve any actual integration.

The localization produced by the delta function turns out to set $\ell_1 = tp_i$ and $\ell_2 = P_L - tp_i$ where $P_L = p_i + \ldots + p_j$ and $t = t_i^{[j-i+1]}/(2p_i \cdot P_L)$. This is shown in appendix B.

Therefore, the result of the action of the collinear operator on the cut is schematically

$$\begin{aligned} &[F_{i,i+1,i+2}, \eta]C_{i,i+1,\ldots,j-1,j} = \\ &\mathcal{J} \times \frac{\langle k \; m\rangle^4}{\langle i \; i+1\rangle \ldots \langle j-1 \; j\rangle} \frac{1}{\langle j \; \ell_2\rangle} A^{\rm tree}(\ell_2, j+1, j+2, \ldots, i-1, \ell_1) \end{aligned} \tag{2.17}$$

with ℓ_1 and ℓ_2 set to the values given above and $\mathcal{J}$ is a Jacobian factor that needs to be computed[10].

This proves that the action of $[F_{i,i+1,i+2}, \eta]$ on $C_{i,i+1,\ldots,j-1,j}$ is a rational function[11].

Important examples of one-loop amplitudes with *all* cuts satisfying the criterion stated above are MHV amplitudes and the next-to-MHV amplitudes $A_{n;1}(1^+, 2^+, 3^+, 4^-, \ldots, n^-)$. In section 3, we compute C_{123} of the latter.

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[9] This formula is not directly applicable to $[F_{i,i+1,i+2}, \eta]C_{i,i+1,\ldots,j-1,j}$. A Schouten identity (2.11) has to applied to go from one to the other. See appendix B for more details.

[10] We have included the factor of $\langle i+1 \; i+2\rangle$ from the collinear operator in the definition of the Jacobian $\mathcal{J}$.

[11] Recall that all tree-level amplitudes are rational functions.

We have shown that the evaluation of certain collinear operators on cuts of the form (2.12) is very simple. As claimed in the introduction, $[F_{i,i+1,i+2},\eta]C_{i,i+1,\ldots,j-1,j}$ is a rational function.

The question is whether this can be used to obtain the explicit form of the cut in terms of the imaginary part of scalar box functions.

The idea is to apply $[F_{i,i+1,i+2},\eta]$ to (2.6), i.e,

$$[F_{i,i+1,i+2},\eta]C_{i,i+1,\ldots,j-1,j} = [F_{i,i+1,i+2},\eta]\,\mathrm{Im}|_{t_i^{[j-i+1]}>0}A_{n;1}. \tag{2.18}$$

with $A_{n:1}$ given by (2.4). Generically, all coefficients of (2.4) are unknown.

It turns out that the imaginary part of any scalar box function is of the form $\ln G$ where F is some rational function of the momentum invariants. To be more precise, this is the case for all scalar box functions except for the four-mass function, where G can have the form $F+\sqrt{K}$, with F and K rational functions (see appendix A). At the end of appendix A, we prove that this rules out all four-mass box functions in cuts of the form (2.12).

The collinear operator (2.9) is a first order differential operator. Therefore, it produces two terms for each term in $\mathrm{Im}|_{t_i^{[j-i+1]}>0}A_{n:1}$. Namely, one term when it acts on the coefficient of a given scalar box function and one more when it acts on the logarithm, i.e., the imaginary part of the scalar box function. When the collinear operator acts on the logarithms, it produces a rational function[12]. On the other hand, when it acts on the coefficients, the logarithm survives.

There are only two ways this can be consistent with the fact that $[F_{i,i+1,i+2},\eta]C_{i,i+1,\ldots,j-1,j}$ must be a rational function. One possibility is that $[F_{i,i+1,i+2},\eta]$ gives zero when acting on the coefficients. The other possibility is that terms coming from different box functions conspire to cancel the logarithms. We now prove that the latter possibility is ruled out.

The proof goes along the same lines as the proof of the cut-constructibility of $\mathcal{N}=4$ amplitudes in [7]. We take the scalar box functions and consider the limit when the only positive kinematical invariant is large. In this limit, the imaginary part of each of the scalar box function develops a unique function of the form

$$\mathrm{Im}|_{t_i^{[r]}>0}\left(\ln(-t_i^{[r]})\ln(-t_{i'}^{[r']})\right) = \pi\ln(-t_{i'}^{[r']}). \tag{2.19}$$

[12] This is where the four-mass function is different from the others. See appendix A.

Upon applying a collinear operator to (2.19), one gets a pole, $1/t_{i'}^{[r']}$, that could be called the "signature" of the corresponding scalar box function in this channel.

Let us study all scalar box functions one at a time. Recall that we only consider cuts in more than two-particle channels (see footnote on page 9).

1. The three-mass box function, $F^{3m}_{n:r:r';i}$. This function participates in four cuts:

$$\begin{aligned}
&\mathrm{Im}\big|_{t_i^{[r]}>0}\Big(\ln(-t_i^{[r]})\ln(-t_{i+r+r'}^{[n-r-r'-1]})\Big) = \pi\ln(-t_{i+r+r'}^{[n-r-r'-1]}),\\
&\mathrm{Im}\big|_{t_{i+r+r'}^{[n-r-r'-1]}>0}\Big(\ln(-t_i^{[r]})\ln(-t_{i+r+r'}^{[n-r-r'-1]})\Big) = \pi\ln(-t_i^{[r]}),\\
&\mathrm{Im}\big|_{t_{i-1}^{[r+1]}>0}\Big(\ln(-t_{i-1}^{[r+1]})\ln(-t_i^{[r+r']})\Big) = \pi\ln(-t_i^{[r+r']}),\\
&\mathrm{Im}\big|_{t_i^{[r+r']}>0}\Big(\ln(-t_{i-1}^{[r+1]})\ln(-t_i^{[r+r']})\Big) = \pi\ln(-t_{i-1}^{[r+1]}).
\end{aligned} \tag{2.20}$$

2. The two-mass "easy" function, $F^{2m\,e}_{n:r;i}$. This function also participates in four cuts:

$$\begin{aligned}
&\mathrm{Im}\big|_{t_i^{[r]}>0}\Big(\ln(-t_i^{[r]})\ln(-t_{i+r+1}^{[n-r-2]})\Big) = \pi\ln(-t_{i+r+1}^{[n-r-2]}),\\
&\mathrm{Im}\big|_{t_{i+r+1}^{[n-r-2]}>0}\Big(\ln(-t_i^{[r]})\ln(-t_{i+r+1}^{[n-r-2]})\Big) = \pi\ln(-t_i^{[r]}),\\
&\mathrm{Im}\big|_{t_{i-1}^{[r+1]}>0}\Big(\ln(-t_{i-1}^{[r+1]})\ln(-t_i^{[r+1]})\Big) = \pi\ln(-t_i^{[r+1]}),\\
&\mathrm{Im}\big|_{t_i^{[r+1]}>0}\Big(\ln(-t_{i-1}^{[r+1]})\ln(-t_i^{[r+1]})\Big) = \pi\ln(-t_{i-1}^{[r+1]}).
\end{aligned} \tag{2.21}$$

3. The two-mass "hard" function, $F^{2m\,h}_{n:r;i}$. This function only participates in three cuts:

$$\begin{aligned}
&\mathrm{Im}\big|_{t_i^{[r]}>0}\Big(\ln(-t_i^{[r]})\ln(-t_{i-1}^{[r+1]})\Big) = \pi\ln(-t_{i-1}^{[r+1]}),\\
&\mathrm{Im}\big|_{t_{i-1}^{[r+1]}>0}\Big(\ln(-t_{i-1}^{[r+1]})\ln(-t_{i-2}^{[2]})\Big) = \pi\ln(-t_{i-2}^{[2]})\\
&\mathrm{Im}\big|_{t_{i+r}^{[n-r-2]}>0}\Big(\ln(-t_{i+r}^{[n-r-2]})\ln(-t_{i-1}^{[r+1]})\Big) = \pi\ln(-t_{i-1}^{[r+1]}).
\end{aligned} \tag{2.22}$$

4. The one-mass function, $F^{1m}_{n;i}$. This function only participates in one cut:

$$\mathrm{Im}\big|_{t_{i-3}^{[3]}>0}\Big(\ln(-t_{i-3}^{[3]})\ln(t_{i-3}^{[2]}t_{i-2}^{[2]})\Big) = \pi\ln(t_{i-3}^{[2]}t_{i-2}^{[2]}). \tag{2.23}$$

By inspection it is clear that the corresponding "signatures" in a given channel are indeed unique.

There is one possible problem with this argument. Suppose that one chooses a collinear operator that does not annihilate a given scalar box function but it annihilates its "signature". In such a case, one has to be more careful and look for other ways to single out the corresponding box function.

In order to make use of these unique signatures, we have to show that there exits at least one collinear operator that is completely safe from the problem mentioned above. Consider any collinear operator made out of gluons in only the left side of the cut. Such operator annihilates the kinematical invariant which is positive and large. It is easy to see that such operator does not annihilate any of the signatures unless it annihilates the whole box function.

This concludes our proof that rules out the possibility of a conspiracy.

Finally, all we have to do is to find out a way of matching the two rational functions, (2.18) and (2.12), in order to extract the coefficients. Clearly, the idea is to look for the "signatures" that appear in (2.18) in the action of $[F_{i,i+1,i+2},\eta]$ on (2.12). We illustrate this procedure in the rest of the paper by computing the $(1,2,3)$ cut of $A_{n:1}(1^+,2^+,3^+,4^-,\ldots,n^-)$.

A Subtlety

As mentioned above, sometimes some of the scalar box functions participating in the cut $C_{i,i+1,\ldots,j-1,j}$ are annihilated by the collinear operator $[F_{i,i+1,i+2},\eta]$. Therefore, no information can be obtained about the corresponding coefficients from this operator.

The way to solve this problem is to apply a different operator that does not annihilate the scalar box functions that the first operator annihilated. The example we consider in the next section exhibits this generic behavior.

3. Computation Of The $t_1^{[3]}$ Cut Of $A_{n:1}^{\mathrm{one-loop}}(1^+,2^+,3^+,4^-,5^-,\ldots,n^-)$

In this section we present the analysis of the $t_1^{[3]}$ cut of the n-gluon one-loop $\mathcal{N}=4$ amplitude $A_{n:1}(1^+,2^+,3^+,4^-,5^-,\ldots,n^-)$. As discussed in the previous section, the full amplitude is a sum over box functions. Here we compute the coefficients of all the box functions that have a branch cut in the $t_1^{[3]}$ channel. In general, there are $(n-2)(n-3)/2$ scalar box functions with cuts in this channel.

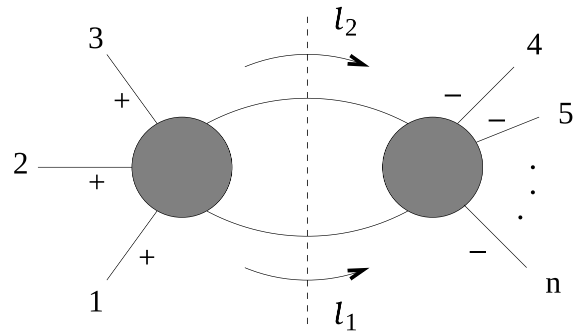

Fig. 2: Cut integral in the $t_1^{[3]}$ channel.

The cut is explicitly given by

$$\begin{aligned} C_{123} &= \int d\mu A^{\text{tree}}((-\ell_1)^-, 1^+, 2^+, 3^+, (-\ell_2)^-) A^{\text{tree}}(\ell_2^+, 4^-, 5^-, \ldots, n^-, \ell_1^+) \\ &= -\frac{1}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\ldots[n-1\ n]} \int d\mu \frac{\langle \ell_1\ \ell_2\rangle^3[\ell_1\ \ell_2]^3}{\langle \ell_1\ 1\rangle\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]} \\ &= -\frac{(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\ldots[n-1\ n]} \int d\mu \frac{1}{\langle \ell_1\ 1\rangle\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}, \end{aligned} \tag{3.1}$$

where $d\mu$ is a measure over the Lorentz invariant phase space of (ℓ_1, ℓ_2), which we write down explicitly below. Also note that in this case, gluons are the only particles that can run in the loop. Moreover, they can only have the helicities given in (3.1). The reason is that one of the tree-level amplitudes in (3.1) is zero for internal scalars, fermions, or gluons with a different helicity assignment (see for example [1]).

If the amplitude was known, C_{123} could be computed by taking the imaginary part of it in the kinematical regime where $t_1^{[3]} > 0$ and all other invariants are negative. More explicitly, the cut would be given by

$$\begin{aligned} C_{123} = \mathrm{Im}|_{t_1^{[3]}>0}\ \Big(& b_4 F^{1m}_{n:4} + c_{2,2} F^{2m\ e}_{n:2;2} + c_{3,1} F^{2m\ e}_{n:3;1} + c_{2,1} F^{2m\ e}_{n:2;1} + d_{2,2} F^{2m\ h}_{n:2;2} + d_{3,1} F^{2m\ h}_{n:3;1} \\ & + d_{n-5,6} F^{2m\ h}_{n:n-5;6} + d_{n-4,5} F^{2m\ h}_{n:n-4;5} + \sum_{r'=2}^{n-5} g_{2,r',2} F^{3m}_{n:2:r';2} + \sum_{r'=2}^{n-6} g_{3,r',1} F^{3m}_{n:3:r';1} \\ & + \sum_{i=7}^{n-1} g_{n-i-1,3,i} F^{3m}_{n:n-i-1:3;i} + \sum_{r=2}^{n-6} g_{r,n-r-4,5} F^{3m}_{n:r:n-r-4;5} + \\ & \sum_{r=2}^{n-4} g_{r,n-r-3,4} F^{3m}_{n:r:n-r-3;4} + \sum_{r'=2}^{n-7} \sum_{r''=2}^{n-r'-5} f_{3,r',r'',1} F^{4m}_{n:3:r':r'';1} \Big) . \end{aligned} \tag{3.2}$$

We have written only the $(n-2)(n-3)/2$ scalar box functions that develop an imaginary part.

The imaginary part of each of these functions can be computed using the formulas given in appendix A.

The main result of this section is the computation of all $(n-2)(n-3)/2$ coefficients in (3.2). The result is strikingly simple and is given by[13]

$$C_{123} = \mathcal{B}_n \ \mathrm{Im}|_{t_1^{[3]}>0} \left(F^{1m}_{n:4} + F^{2m\ e}_{n:3;1} + F^{2m\ h}_{n:2;2} + F^{2m\ h}_{n:n-4;5} \right) \tag{3.3}$$

where

$$\mathcal{B}_n = \frac{(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\ldots[n-1\ n]\langle 1|1+2+3|4]\langle 3|1+2+3|n]} \tag{3.4}$$

and $\langle i|1+2+3|k] = \langle i\ 1\rangle[1\ k] + \langle i\ 2\rangle[2\ k] + \langle i\ 3\rangle[3\ k]$.

3.1. Collinear Operators Acting On C_{123}

The explicit form of C_{123} in (3.1) makes it manifest that its twistor support has gluons $1, 2, 3$ on the same "line" or $\mathbb{CP}^1$, i.e., they are collinear. This is because mostly plus tree-level MHV amplitudes are localized on a line [9]. If there was no holomorphic anomaly, this localization would imply that the differential operator $[F_{123}, \eta]$ annihilates C_{123}. According to the general discussion of section 2, this is precisely the kind of operators that can give the most information about the cut.

Let us compute $[F_{123}, \eta]C_{123}$ explicitly using the holomorphic anomaly. The computation starts by applying $[F_{123}, \eta]$ to C_{123} given in (3.1). As expected, we find that the operator only acts on the poles,

$$\begin{aligned} &[F_{123}, \eta]C_{123} = \frac{(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\ldots[n-1\ n]} \times \\ &\int d\mu \left[\frac{1}{\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}\langle 2\ 3\rangle[\partial_1, \eta]\left(\frac{1}{\langle \ell_1\ 1\rangle}\right) + \frac{1}{\langle \ell_1\ 1\rangle[\ell_2\ 4][n\ \ell_1]}\langle 1\ 2\rangle[\partial_3, \eta]\left(\frac{1}{\langle 3\ \ell_2\rangle}\right)\right]. \end{aligned} \tag{3.5}$$

To see this note that $[F_{123}, \eta]t_1^{[3]} = 0$ due to (2.11). Recall that $\partial_k = \partial/\partial\widetilde{\lambda}_k$.

Let us concentrate on the computation of the first term leaving out the overall factor containing $t_1^{[3]}$. In order to avoid cluttering the formulas, it is convenient to introduce

$$P = p_1 + p_2 + p_3. \tag{3.6}$$

[13] Recall that we are omitting all constants that appear as overall factors, including factors of i and 2π. These can be easily worked out if needed.

We have to write the Lorentz invariant measure explicitly

$$\int d\mu \frac{1}{\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}\langle 2\ 3\rangle[\partial_1,\eta]\left(\frac{1}{\langle \ell_1\ 1\rangle}\right) =$$
$$\int d^4\ell_1 \delta^{(+)}(\ell_1^2)\int d^4\ell_2\delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)\frac{1}{\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}\langle 2\ 3\rangle[\partial_1,\eta]\left(\frac{1}{\langle \ell_1\ 1\rangle}\right). \tag{3.7}$$

As mentioned above, counting the two delta functions produced by the holomorphic anomaly, i.e., by the action of $[\partial_1,\eta]$ on the pole, we have a total of eight delta functions. These are enough to localize the integral (3.7) completely, i.e., no actual integration has to be performed.

In appendix B, we provide a detailed computation of (3.7). Let us summarize the results of the localization: $\ell_1 = tp_1$, $\ell_2 = P - tp_1$, with $t = P^2/(2p_1\cdot P)$. The evaluation in appendix B shows that (3.7) is

$$\int d\mu \frac{1}{\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}\langle 2\ 3\rangle[\partial_1,\eta]\left(\frac{1}{\langle \ell_1\ 1\rangle}\right) = \frac{[1\ \eta]}{\langle 1|P|4][n\ 1][1\ 2]}. \tag{3.8}$$

This is essentially the general form given in (2.17) with the jacobian explicitly computed.

The computation of the second term in (3.5) is completely similar. Combining the two results, we obtain the final form for the action of the collinear operator on the cut

$$[F_{123},\eta]C_{123} = \frac{(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\ldots[n-1\ n]}\left(\frac{[1\ \eta]}{\langle 1|P|4][n\ 1][1\ 2]}+\frac{[3\ \eta]}{\langle 3|P|n][2\ 3][3\ 4]}\right). \tag{3.9}$$

"Simple Fraction" Expansion

Although our formula (3.9) for $[F_{123},\eta]C_{123}$ is quite explicit, it is still not enough. Recall that the goal is to compare (3.9) to the action of $[F_{123},\eta]$ on the imaginary part of scalar box functions. As discussed in section 2 and more fully in appendix A, these imaginary parts are logarithms of simple functions of the kinematical invariants. Once the collinear operator is applied, it produces sums over different poles which we loosely call "simple fractions".

Therefore, we have to expand (3.9) in "simple fractions". The precise meaning of this will become clear as we do it.

Note that the overall coefficient of (3.9) is not relevant as it is annihilated by $[F_{123},\eta]$, so we have to concentrate on the terms inside the parenthesis. Moreover, $[F_{123},\eta]$ also annihilates $\langle 1|P|4]$ and $\langle 3|P|n]$, so we have to factor them out

$$[F_{123},\eta]C_{123} = \frac{(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\dots[n-1\ n]\langle 1|P|4]\langle 3|P|n]}\left(\frac{[1\ \eta]\langle 3|P|n]}{[n\ 1][1\ 2]}+\frac{[3\ \eta]\langle 1|P|4]}{[2\ 3][3\ 4]}\right). \tag{3.10}$$

Recall that $P=p_1+p_2+p_3$. Using Schouten identity (2.11), we find

$$\left(\frac{[1\ \eta]\langle 3|P|n]}{[n\ 1][1\ 2]}+\frac{[3\ \eta]\langle 1|P|4]}{[2\ 3][3\ 4]}\right) = \langle 1\ 2\rangle\frac{\langle 3|P|\eta]}{t_1^{[2]}}-\langle 2\ 3\rangle\frac{\langle 1|n|\eta]}{t_n^{[2]}}+\langle 2\ 3\rangle\frac{\langle 1|P|\eta]}{t_2^{[2]}}-\langle 1\ 2\rangle\frac{\langle 3|4|\eta]}{t_3^{[2]}} \tag{3.11}$$

Note that

$$\begin{aligned}\langle 1\ 2\rangle\langle 3|P|\eta] &= [F_{123},\eta]\left(t_1^{[2]}\right), \quad \langle 2\ 3\rangle\langle 1|P|\eta] = [F_{123},\eta]\left(t_n^{[2]}\right),\\ \langle 2\ 3\rangle\langle 1|P|\eta] &= [F_{123},\eta]\left(t_2^{[2]}\right), \quad \langle 1\ 2\rangle\langle 3|4|\eta] = [F_{123},\eta]\left(t_3^{[2]}\right).\end{aligned} \tag{3.12}$$

Finally, it is easy to identify each term in (3.11) by using (3.12) with the action of $[F_{123},\eta]$ on the logarithm of $t_i^{[2]}$ for some i. Combining all logarithms into one we find

$$[F_{123},\eta]C_{123} = \frac{(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\dots[n-1\ n]\langle 1|P|4]\langle 3|P|n]}[F_{123},\eta]\log\left(\frac{t_1^{[2]}t_2^{[2]}}{t_3^{[2]}t_n^{[2]}}\right). \tag{3.13}$$

3.2. Collinear Operator Acting On Box Functions

In this section we compute the action of $[F_{123},\eta]$ on C_{123} using the formula in terms of the imaginary part of box functions (3.2)

$$\begin{aligned}C_{123} = \mathrm{Im}|_{t_1^{[3]}>0}\ \Big(& b_4F^{1m}_{n:4}+c_{2,2}F^{2m\ e}_{n:2;2}+c_{3,1}F^{2m\ e}_{n:3;1}+c_{2,1}F^{2m\ e}_{n:2;1}+d_{2,2}F^{2m\ h}_{n:2;2}+d_{3,1}F^{2m\ h}_{n:3;1}\\ &+d_{n-5,6}F^{2m\ h}_{n:n-5;6}+d_{n-4,5}F^{2m\ h}_{n:n-4;5}+\sum_{r'=2}^{n-5}g_{2,r',2}F^{3m}_{n:2:r';2}+\sum_{r'=2}^{n-6}g_{3,r',1}F^{3m}_{n:3:r';1}\\ &+\sum_{i=7}^{n-1}g_{n-i-1,3,i}F^{3m}_{n:n-i-1:3;i}+\sum_{r=2}^{n-6}g_{r,n-r-4,5}F^{3m}_{n:r:n-r-4;5}+\\ &\sum_{r=2}^{n-4}g_{r,n-r-3,4}F^{3m}_{n:r:n-r-3;4}+\sum_{r'=2}^{n-7}\sum_{r''=2}^{n-r'-5}f_{3,r',r'',1}F^{4m}_{n:3:r':r'';1}\Big).\end{aligned} \tag{3.14}$$

Now we find the subtlety discussed at the end of section 2. It turns out that $[F_{123}, \eta]$ annihilates each of the scalar box functions in the following sum

$$c_{3,1}F^{2m\ e}_{n:3;1} + d_{3,1}F^{2m\ h}_{n:3;1} + d_{n-5,6}F^{2m\ h}_{n:n-5;6} + \sum_{r'=2}^{n-6} g_{3,r',1}F^{3m}_{n:3:r';1} + \sum_{i=7}^{n-1} g_{n-i-1,3,i}F^{3m}_{n:n-i-1:3;i}$$
$$+ \sum_{r=2}^{n-6} g_{r,n-r-4,5}F^{3m}_{n:r:n-r-4;5} + \sum_{r'=2}^{n-7}\sum_{r''=2}^{n-r'-5} f_{3,r',r'',1}F^{4m}_{n:3:r':r'';1}. \tag{3.15}$$

From their explicit form given in appendix B, it is easy to see that each of them depends on gluons 1, 2, and 3 only through $p_1 + p_2 + p_3$. By the discussion that led to (2.11) this is condition is enough to ensure total annihilation by $[F_{123}, \eta]$.

Therefore, no information can be obtained about these coefficients with this collinear operator. All this implies is that in order to get the whole cut we have to consider the action of at least one more collinear operator. We do this in section 3.3.

After removing the terms in (3.15), we are left with

$$[F_{123}, \eta]C_{123} = [F_{123}, \eta] \,\mathrm{Im}|_{t_1^{[3]}>0} \big(b_4F^{1m}_{n:4} + c_{2,2}F^{2m\ e}_{n:2;2} + c_{2,1}F^{2m\ e}_{n:2;1} + d_{2,2}F^{2m\ h}_{n:2;2}$$
$$+ d_{n-4,5}F^{2m\ h}_{n:n-4;5} + \sum_{r'=2}^{n-5} g_{2,r',2}F^{3m}_{n:2:r';2} + \sum_{r=2}^{n-4} g_{r,n-r-3,4}F^{3m}_{n:r:n-r-3;4}\Bigg). \tag{3.16}$$

The imaginary parts of the box functions can be easily obtained as explained in appendix A. Here we will concentrate first on those scalar box function which produce a term that is not present in (3.13).

Consider for example,

$$\mathrm{Im}|_{t_1^{[3]}>0}F^{2m\ e}_{n:2;2} = \pi \ln\left(1 - \frac{t_2^{[2]}t_1^{[4]}}{t_1^{[3]}t_2^{[3]}}\right) + \ldots \tag{3.17}$$

Upon acting with the collinear operator this produces a contribution to (3.16) of the form

$$[F_{123}, \eta] \ln(t_2^{[2]}t_1^{[4]} - t_1^{[3]}t_2^{[3]}) \tag{3.18}$$

which is non zero and it is not present in (3.13). Therefore we conclude that $c_{2,2} = 0$.

One might wonder whether there are other terms in (3.16) that might cancel this contribution so that the equation $c_{2,2} = 0$ is be replaced by a relation between several coefficients. As discussed in section 2, this is not the case, for each box function has a "unique signature".

Consider the limit when $t_1^{[3]}$ is large. Then $\ln(t_2^{[2]}t_1^{[4]} - t_1^{[3]}t_2^{[3]})$ produces a term of the form $\ln(-t_2^{[3]})$. Looking at (2.21) we see that this is the signature of $F^{2m\ e}_{n:2;2}$ in the $t_1^{[3]}$ channel. In other words, there is no other box function that could produce such a term.

The same analysis can be repeated to find that $c_{2,1} = g_{2,r',2} = g_{r,n-r-3,4} = 0$ for all r and r' in the sums given in (3.16).

Let us study in detail the imaginary parts of the three box functions with nonzero coefficients remaining in (3.16)

$$
\begin{aligned}
\mathrm{Im}|_{t_1^{[3]}>0}\, F^{1m}_{n:4} &= -\ln\left(1-\frac{t_1^{[3]}}{t_1^{[2]}}\right) - \ln\left(1-\frac{t_1^{[3]}}{t_2^{[2]}}\right);\\
\mathrm{Im}|_{t_1^{[3]}>0}\, F^{2m\ h}_{n:2;2} &= \ln\left(-\frac{t_1^{[3]}}{t_n^{[2]}}\right) + \ln\left(1-\frac{t_2^{[2]}}{t_1^{[3]}}\right) + \ln\left(1-\frac{t_n^{[4]}}{t_1^{[3]}}\right);\\
\mathrm{Im}|_{t_1^{[3]}>0}\, F^{2m\ h}_{n:n-4;5} &= \ln\left(-\frac{t_1^{[3]}}{t_3^{[2]}}\right) + \ln\left(1-\frac{t_5^{[n-4]}}{t_1^{[3]}}\right) + \ln\left(1-\frac{t_1^{[2]}}{t_1^{[3]}}\right).
\end{aligned}
\tag{3.19}
$$

The problem at hand is to find $b_4, d_{2,2}$ and $d_{n-4,5}$ such that (3.16) equals (3.13). More explicitly, we need

$$
\begin{aligned}
&[F_{123},\eta]\ \mathrm{Im}|_{t_1^{[3]}>0}\left(b_4F^{1m}_{n:4} + d_{2,2}F^{2m\ h}_{n:2;2} + d_{n-4,5}F^{2m\ h}_{n:n-4;5}\right) = \\
&\qquad \frac{(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\ldots[n-1\ n]\langle 1|P|4]\langle 3|P|n]}[F_{123},\eta]\ln\left(\frac{t_1^{[2]}t_2^{[2]}}{t_3^{[2]}t_n^{[2]}}\right).
\end{aligned}
\tag{3.20}
$$

First note that the imaginary part of $F^{1m}_{n:4}$ is the only one that contains the term $\ln(t_1^{[2]}t_2^{[2]})$, again by looking at (2.23) we find that this is the signature of $F^{1m}_{n:4}$ in this channel. On the other hand, $F^{2m\ h}_{n:2;2}$ is the only function that produces $\ln(-t_n^{[2]})$ and finally, $\ln(-t_3^{[2]})$ is unique to the imaginary part of $F^{2m\ h}_{n:n-4;5}$. This implies that

$$
b_4 = d_{2,2} = d_{n-4,5} = \frac{(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\ldots[n-1\ n]\langle 1|P|4]\langle 3|P|n]}. \tag{3.21}
$$

Indeed, setting all coefficients equal and realizing that

$$
[F_{123},\eta]\ \ln\left(1-\frac{t_n^{[4]}}{t_1^{[3]}}\right) = 0, \qquad [F_{123},\eta]\ \ln\left(1-\frac{t_5^{[2]}}{t_1^{[3]}}\right) = 0, \tag{3.22}
$$

one can easily check that

$$
[F_{123},\eta]\ \mathrm{Im}|_{t_1^{[3]}>0}\left(F^{1m}_{n:4} + F^{2m\ h}_{n:2;2} + F^{2m\ h}_{n:n-4;5}\right) = [F_{123},\eta]\ln\left(\frac{t_1^{[2]}t_2^{[2]}}{t_3^{[2]}t_n^{[2]}}\right). \tag{3.23}
$$

3.3. Computing The Remaining Coefficients

As mentioned above, the coefficients in (3.15) remain unknown and have to be determined by the action of a different collinear operator. From looking at (3.1) we see that no other set of gluons is manifestly localized on a line. To see this note that the second tree-level amplitude in (3.1) is a mostly minus MHV amplitude and therefore it has gluons localized on a degree $n-4$ curve.

The solution to this problem is clear, consider $C_{123}^{\dagger}$ instead of C_{123}. Then the mostly minus tree-level amplitude becomes a mostly plus MHV amplitude which is localized on a line. Now we can consider the action of $[F_{4in}, \eta]$, with i taking any value from 5 to $n-1$, on $C_{123}^{\dagger}$.

It turns out that $[F_{4in}, \eta]$ only annihilates $F_{n:4}^{1m}$. Therefore, this analysis does not provide information about $b_4^{\dagger}$. However, b_4 is already known (3.21).

Let us start again by writing $C_{123}^{\dagger}$ explicitly

$$\begin{aligned} C_{123}^{\dagger} &= \int d\mu A^{\mathrm{tree}}((-\ell_1)^+, 1^-, 2^-, 3^-, (-\ell_2)^+) A^{\mathrm{tree}}(\ell_2^-, 4^+, 5^+, \ldots, n^+, \ell_1^-) \\ &= -\frac{1}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle \ldots \langle n-1\ n\rangle} \int d\mu \frac{[\ell_1\ \ell_2]^3 \langle \ell_1\ \ell_2\rangle^3}{[\ell_1\ 1][3\ \ell_2]\langle \ell_2\ 4\rangle\langle n\ \ell_1\rangle} \\ &= -\frac{(t_1^{[3]})^3}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle \ldots \langle n-1\ n\rangle} \int d\mu \frac{1}{[\ell_1\ 1][3\ \ell_2]\langle \ell_2\ 4\rangle\langle n\ \ell_1\rangle} \end{aligned} \quad . \tag{3.24}$$

Following exactly the same steps as for $[F_{123}, \eta]C_{123}$, we find

$$[F_{4in}, \eta]C_{123}^{\dagger} = \frac{(t_1^{[3]})^3}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle \ldots \langle n-1\ n\rangle\langle 4|P|1]\langle n|P|3]} \left(\langle 4\ i\rangle \frac{[n\ \eta]\langle 4|P|1]}{[n\ 1]\langle 4|P|n]} + \langle i\ n\rangle \frac{[4\ \eta]\langle n|P|3]}{[4\ 3]\langle n|P|4]} \right). \tag{3.25}$$

Using that $P = p_1 + p_2 + p_3$ and several Schouten's identity we can expand the term in parenthesis in "simple fractions"

$$\langle 4\ i\rangle \frac{\langle 4|P|\eta]}{\langle 4|P|n]} - \langle 4\ i\rangle \frac{[\eta\ 1]}{[n\ 1]} + \langle i\ n\rangle \frac{\langle n|P|\eta]}{\langle n|P|4]} - \langle i\ n\rangle \frac{[\eta\ 3]}{[4\ 3]}. \tag{3.26}$$

In this form, it is easy to identify (3.26) with

$$[F_{4in}, \eta] \ln \left(\frac{\langle 4|P|n]\langle n|P|4]}{[n\ 1][4\ 3]} \right) = -[F_{4in}, \eta] \ln \left(\frac{t_n^{[2]} t_3^{[2]}}{t_n^{[4]} t_1^{[4]} - t_1^{[3]} t_n^{[5]}} \right). \tag{3.27}$$

Combining (3.27) with (3.25) we finally find

$$[F_{4in},\eta]C^{\dagger}_{123} = \frac{(t_1^{[3]})^3}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\ldots\langle n-1\ n\rangle\langle 4|P|1]\langle n|P|3]}[F_{4in},\eta]\ln\left(\frac{t_n^{[2]}t_3^{[2]}}{t_n^{[4]}t_1^{[4]}-t_1^{[3]}t_n^{[5]}}\right). \tag{3.28}$$

As in the analysis of section 3.2, we have to compare (3.28) to the action of $[F_{4in},\eta]$ on the imaginary part of the box functions in (3.14). Recall that (3.28) is supposed to provide information about the coefficients that remained unknown after the study of $[F_{123},\eta]C_{123}$, i.e., the coefficients in (3.15).

By computing the imaginary part of the box functions we see that in order to reproduce (3.28) we have to impose that $d_{3,1}$, $d_{n-5,6}$, $g_{3,r',1}$, $g_{n-i-1,3,i}$, $g_{r,n-r-4,5}$ and $f_{3,r',r'',1}$ all vanish. Thus, the only coefficient left in (3.15) is $c_{3,1}$.

Let us write down the imaginary part of $F^{2m\ e}_{n:3;1}$ explicitly,

$$\mathrm{Im}|_{t_1^{[3]}>0}\ F^{2m\ e}_{n:3;1} = -\ln\left(1-\frac{t_1^{[3]}}{t_n^{[4]}}\right) - \ln\left(1-\frac{t_1^{[3]}}{t_1^{[4]}}\right) + \ln\left(1-\frac{t_1^{[3]}t_n^{[5]}}{t_n^{[4]}t_1^{[4]}}\right). \tag{3.29}$$

From this we can see that this contributes a factor of $\ln(t_n^{[4]}t_1^{[4]}-t_1^{[3]}t_n^{[5]})$ and therefore $c^{\dagger}_{3,1}$ must equal the overall factor in (3.28).

By comparing the overall factor of (3.20) with that of (3.28) we find that one is the complex conjugate of the other. Therefore, using (3.21) we find

$$\begin{aligned}[F_{4in},\eta]C^{\dagger}_{123} = &\frac{(t_1^{[3]})^3}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\ldots\langle n-1\ n\rangle\langle 4|P|1]\langle n|P|3]}\times \\ &[F_{4in},\eta]\mathrm{Im}|_{t_1^{[3]}>0}\left(F^{2m\ h}_{n:2;2}+F^{2m\ h}_{n:n-4;5}+F^{2m\ e}_{m:3;1}\right).\end{aligned} \tag{3.30}$$

We now explicitly check that (3.30) equals (3.28). Even though this is guaranteed to work, it is still interesting to see the interplay between the different imaginary parts.

Using the explicit formulas for the imaginary parts of the box functions involved in (3.30) given in (3.19) and (3.29) we find that (3.30) equals

$$\begin{aligned}[F_{4in},\eta]C^{\dagger}_{123} \ = &\frac{(t_1^{[3]})^3}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\ldots\langle n-1\ n\rangle\langle 4|P|1]\langle n|P|3]}\times \\ &[F_{4in},\eta]\mathrm{Im}|_{t_1^{[3]}>0}\left(\ln\left(\frac{t_n^{[2]}t_3^{[2]}}{t_n^{[4]}t_1^{[4]}-t_1^{[3]}t_n^{[5]}}\right)+\ln\left(1-\frac{t_2^{[2]}}{t_1^{[3]}}\right)+\ln\left(1-\frac{t_1^{[2]}}{t_1^{[3]}}\right)\right).\end{aligned} \tag{3.31}$$

Note that the last two terms in (3.31) are trivially annihilated by $[F_{4in},\eta]$ providing the desired result.

3.4. Summary Of Results

Let us collect all the results we have obtained in order to write down the final formula for the cut C_{123}.

Recall that the cut was given by (3.14)

$$\begin{aligned} C_{123} = \mathrm{Im}|_{t_1^{[3]}>0} \Big(& b_4 F^{1m}_{n:4} + c_{2,2} F^{2m\ e}_{n:2;2} + c_{3,1} F^{2m\ e}_{n:3;1} + c_{2,1} F^{2m\ e}_{n:2;1} + d_{2,2} F^{2m\ h}_{n:2;2} + d_{3,1} F^{2m\ h}_{n:3;1} \\ & + d_{n-5,6} F^{2m\ h}_{n:n-5;6} + d_{n-4,5} F^{2m\ h}_{n:n-4;5} + \sum_{r'=2}^{n-5} g_{2,r',2} F^{3m}_{n:2:r';2} + \sum_{r'=2}^{n-6} g_{3,r',1} F^{3m}_{n:3:r';1} \\ & + \sum_{i=7}^{n-1} g_{n-i-1,3,i} F^{3m}_{n:n-i-1:3;i} + \sum_{r=2}^{n-6} g_{r,n-r-4,5} F^{3m}_{n:r:n-r-4;5} + \\ & \sum_{r=2}^{n-4} g_{r,n-r-3,4} F^{3m}_{n:r:n-r-3;4} + \sum_{r'=2}^{n-7} \sum_{r"=2}^{n-r'-5} f_{3,r',r'',1} F^{4m}_{n:3:r':r'';1} \Big) . \end{aligned} \tag{3.32}$$

It is convenient to define

$$\mathcal{B}_n = \frac{(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\ldots[n-1\ n]\langle 1|P|4]\langle 3|P|n]}. \tag{3.33}$$

From the analysis of $[F_{123}, \eta]C_{123}$ we found

$$\begin{aligned} & b_4 = d_{2,2} = d_{n-4,5} = \mathcal{B}_n, \\ & c_{2,2} = c_{2,1} = g_{2,r',2} = g_{r,n-r-3,4} = 0. \end{aligned} \tag{3.34}$$

And from the analysis of $[F_{4in}, \eta]C^\dagger_{123}$ we found

$$\begin{aligned} & c_{3,1} = \mathcal{B}_n, \\ & d_{3,1} = d_{n-5,6} = g_{3,r',1} = g_{n-i-1,3,i} = g_{r,n-r-4,5} = f_{3,r',r'',1} = 0. \end{aligned} \tag{3.35}$$

Putting together (3.34) and (3.35) we have all coefficients appearing in (3.32). The final result for the cut is then

$$C_{123} = \mathcal{B}_n\ \mathrm{Im}|_{t_1^{[3]}>0} \left(F^{1m}_{n:4} + F^{2m\ e}_{n:3;1} + F^{2m\ h}_{n:2;2} + F^{2m\ h}_{n:n-4;5} \right). \tag{3.36}$$

4. Consistency Checks

In this section, we study several consistency checks that (3.36) must satisfy. The first is the requirement that the cut must be free of the IR divergencies that show up in the scalar box functions. The others are just the comparison of (3.36) with the cuts of known amplitudes, i.e., $n = 4, 5$ and 6.

4.1. Finiteness Of The Cut

All box functions, except for the four-mass, are infrared divergent. This means that they have to be evaluated using some regularization procedure. The explicit formulas in appendix A are in the dimensional regularization scheme. On the other hand, the $(1,2,3)$-cut we have computed has to be finite. The reason is that, as mentioned in section 2, the integration region is compact; it is a two-sphere. Therefore, the divergencies of the imaginary part of each of the box functions appearing in (3.36) must cancel out.

The infrared divergent structure of each of these function is given by

$$\begin{aligned}
F^{1m}_{n:4}|_{\rm IR} &= -\frac{1}{\epsilon^2}\left[(-t_1^{[2]})^{-\epsilon} + (-t_2^{[2]})^{-\epsilon} - (-t_1^{[3]})^{-\epsilon}\right];\\
F^{2m\ e}_{n:3;1}|_{\rm IR} &= -\frac{1}{\epsilon^2}\left[(-t_n^{[4]})^{-\epsilon} + (-t_1^{[4]})^{-\epsilon} - (-t_1^{[3]})^{-\epsilon} - (-t_n^{[5]})^{-\epsilon}\right];\\
F^{2m\ h}_{n:2;2}|_{\rm IR} &= -\frac{1}{\epsilon^2}\left[\frac{1}{2}(-t_n^{[2]})^{-\epsilon} + (-t_1^{[3]})^{-\epsilon} - \frac{1}{2}(-t_1^{[2]})^{-\epsilon} - \frac{1}{2}(-t_n^{[4]})^{-\epsilon}\right];\\
F^{2m\ h}_{n:n-4;5}|_{\rm IR} &= -\frac{1}{\epsilon^2}\left[\frac{1}{2}(-t_3^{[2]})^{-\epsilon} + (-t_4^{[n-3]})^{-\epsilon} - \frac{1}{2}(-t_5^{[n-4]})^{-\epsilon} - \frac{1}{2}(-t_1^{[2]})^{-\epsilon}\right].
\end{aligned} \tag{4.1}$$

It is clear that the only term in each of these functions that develops an imaginary part is $(-t_1^{[3]})^{-\epsilon}$. Note that in the sum they cancel out. To see this more clearly, recall that by momentum conservation $t_4^{[n-3]} = t_1^{[3]}$.

4.2. Four- and Five-Gluon Amplitudes

Taking $n=4$ we find that $\mathcal{B}_n$ is trivially zero. To see this note that $t_1^{[3]} = p_4^2 = 0$. The zeroes in the denominator are cancelled by some of the three powers of $t_1^{[3]}$ in the numerator. This is consistent with the fact that the amplitude $A_{4:1}(1^+,2^+,3^+,4^-)$ is exactly zero.

More interesting is the case when $n=5$. This amplitude was first computed in [24]. Setting $n=5$ in the definition of $\mathcal{B}_n$ we find that

$$\mathcal{B}_5 = \frac{\langle 4\ 5\rangle^3}{\langle 5\ 1\rangle\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle}. \tag{4.2}$$

This is the-tree level MHV amplitude $A^{\rm tree}(1^+,2^+,3^+,4^-,5^-)$. Also note that for $n=5$ the last three box functions in (3.36) naturally descend to one-mass scalar box functions as follows: $F^{2m\ e}_{5:3;1} = 0$, $F^{2m\ h}_{5:2;2} = F^{1m}_{5:2}$ and $F^{2m\ h}_{5:1;5} = F^{1m}_{5;1}$. This gives

$$C_{123} = A^{\rm tree}(1^+,2^+,3^+,4^-,5^-)\ {\rm Im}|_{t_1^{[3]}>0}\left(F^{1m}_{5:4} + F^{1m}_{5:2} + F^{1m}_{5;1}\right) \tag{4.3}$$

which is the right answer for the cut.

4.3. Six-Gluon Amplitude

The one-loop non-MHV six gluon partial amplitude $A_{6;1}^{\mathcal{N}=4}(1^+,2^+,3^+,4^-,5^-,6^-)$ was first computed by Bern et.al. in [8]. They computed C_{123} by different methods to the ones presented here and obtained

$$C_{123} = B_0 \ \mathrm{Im}|_{t_1^{[3]}>0} \left(F_{6:1}^{1m} + F_{6:4}^{1m} + F_{6:2;2}^{2m\ h} + F_{6:2;5}^{2m\ h} \right). \tag{4.4}$$

where

$$B_0 = \frac{([1\ 2]\langle 2\ 4\rangle + [1\ 3]\langle 3\ 4\rangle)([3\ 1]\langle 1\ 6\rangle + [3\ 2]\langle 2\ 6\rangle)(t_1^{[3]})^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6](t_1^{[3]}t_3^{[3]} - t_1^{[2]}t_4^{[2]})(t_1^{[3]}t_2^{[3]} - t_2^{[2]}t_5^{[2]})}. \tag{4.5}$$

In order to show that (4.4) is equal to (3.36) with $n=6$ all we have to do is to realize that

$$\begin{aligned} F_{6:3;1}^{2m\ e} &= F_{6:1}^{1m} \\ t_1^{[3]}t_3^{[3]} - t_1^{[2]}t_4^{[5]} &= \langle 6|P|3]\langle 3|P|6] = ([3\ 1]\langle 1\ 6\rangle + [3\ 2]\langle 2\ 6\rangle)\langle 3|P|6] \\ t_1^{[3]}t_2^{[3]} - t_2^{[2]}t_5^{[2]} &= \langle 4|P|1]\langle 1|P|4] = ([1\ 2]\langle 2\ 4\rangle + [1\ 3]\langle 3\ 4\rangle)\langle 1|P|4]. \end{aligned} \tag{4.6}$$

This makes it manifest that B_0 in (4.5) is equal to our $\mathcal{B}_6$ in (3.33).

Something worth mentioning is that all other independent cuts are given in terms of C_{123} [8]. In our notation, C_{234} is given as follows,

$$C_{234} = \left(\frac{[2\ 3]\langle 5\ 6\rangle}{t_{234}}\right)^4 \times \left[C_{123}^\dagger\right]\Big|_{j\to j+1} + \left(\frac{\langle 1|P|4]}{t_{234}}\right)^4 \times \left[C_{123}\right]\big|_{j\to j+1} \tag{4.7}$$

The remaining cut, i.e., C_{345} can be obtained from (4.7) by shifting the labels and conjugation.

Knowing all cuts implies that we know all coefficients in the amplitude and thus the amplitude itself. For the explicit form of the amplitude see [8].

5. Discussion

We have shown that certain unitarity cuts can be determined in terms of scalar box functions in a simple way. The class of cuts for which the method proposed here works involves all the cuts of two interesting series of amplitudes. Namely, the n-gluon one-loop MHV amplitudes and the n-gluon one-loop next-to-MHV amplitudes with three *consecutive* plus helicity gluons and the rest negative. We have checked that our method reproduces correctly all MHV amplitudes. We have not included the computation here because it is very similar to that of our main example.

We have explicitly computed one of the cuts in the next-to-MHV series to show how efficient this method can be. It is important to remark that for $n = 6$ the $(1, 2, 3)$-cut was computed about ten years ago in a pioneering work by Bern, et.al. [8]. That computation uses very powerful reduction techniques [25]. However, it leads to quite complicated formulas before it can be put in the final simple form. Notably, the six-gluon amplitude computed in [8] is actually the only one-loop non-MHV amplitude known to date.

Although we have explicitly computed only one of the cuts of the next-to-MHV series, the computation of the remaining cuts should be within reasonable reach. This implies that the computation of the corresponding series of amplitudes is also within reach.

Even more interesting is the possibility of extending this method to all possible cuts. Clearly, this would have to involve more than just collinear operators since, in general, both tree-level amplitudes appearing in the cut will be non-MHV. However, in [11] a prescription was given to compute all tree-level amplitudes in terms of MHV diagrams. These diagrams are made out of MHV amplitudes continued off-shell and connected by Feynman propagators. In twistor space this corresponds to configurations where gluons are localized on unions of lines. Moreover, from the viewpoint of the differential operators (see section 2 of [12] for more details), these lines intersect to form quivers or trees. So, it is conceivable that by combining collinear operators with coplanar operators (that tests whether four gluons are contained in a common plane in twistor space), one could compute all cuts. Although coplanar operators have not been discussed in the context of the holomorphic anomaly, they are indeed affected by it.

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Appendix A. Scalar Box Functions And Monodromies

Scalar Box functions are a set of functions constructed from the scalar box integrals. The latter form a complete set of the possible integrals that can appear in a Feynman diagrammatic computation of one-loop amplitudes in $\mathcal{N} = 4$ gauge theory.[14]

These integrals are known as the scalar box integrals because they would arise in a one-loop computation of a scalar field theory with four internal propagators.

[14] After Passarino-Veltman reduction formulas are applied.

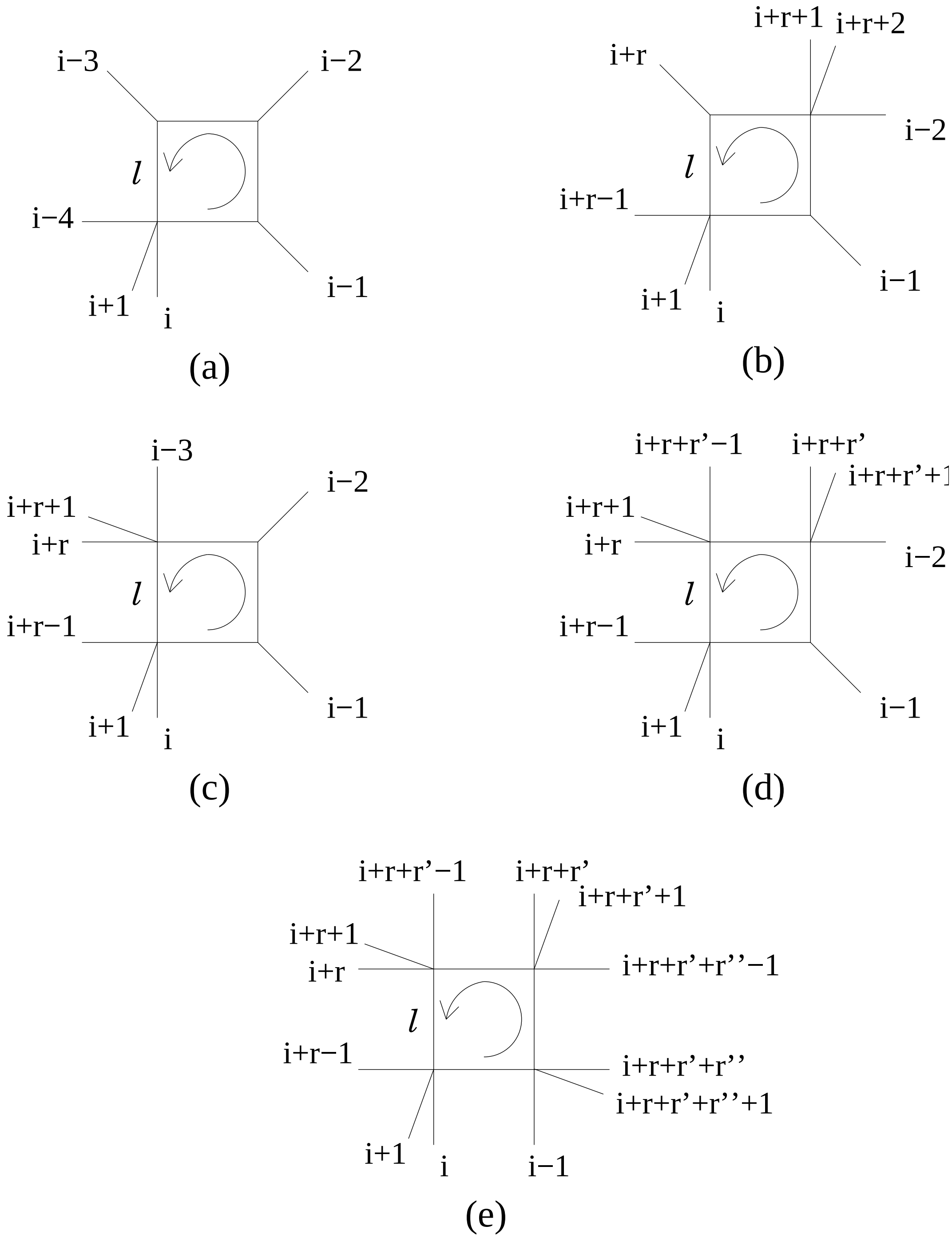

Fig. 3: Scalar box integrals used in the definition of scalar box functions: (a) One-mass $F^{1m}_{n:i}$. (b) Two-mass "easy" $F^{2m\ e}_{n:r;i}$. (c) Two-mass "hard" $F^{2m\ h}_{n:r;i}$. (d) Three-mass $F^{3m}_{n:r:r';i}$. (e) Four-mass $F^{4m}_{n:r:r':r";i}$.

The scalar box integral is defined as follows:

$$I_4 = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2(\ell-K_1)^2(\ell-K_1-K_2)^2(\ell+K_4)^2}. \tag{A.1}$$

The incoming external momenta at each of the vertices are K_1, K_2, K_3, K_4. The labels are given in consecutive order following the loop. Momentum conservation implies that $K_1 + K_2 + K_3 + K_4 = 0$ and this is why (A.1) only depends on three momenta.

In the computation of one-loop $\mathcal{N} = 4$ amplitudes, each of the external momenta becomes a sum of the momenta of external gluons. The evaluation of the integral (A.1) varies in complexity depending upon the number of legs that have $K_i^2 = 0$, i.e., K_i is just a single gluon momentum. The convention is to label the integrals according to the number of $K_i^2 \neq 0$, the so-called "massive legs". In general, the number of massive legs goes from 1 to 4. For two masses there are two inequivalent choices of the massive legs and are called "easy" and "hard". The names refer to how difficult the evaluation of (A.1) is compared to the other. The different possibilities, as shown in fig. 3, are:

$$I_{4:i}^{1m},\ I_{n:r;i}^{2m\ e},\ I_{4:r;i}^{2m\ h},\ I_{4:r:r';i}^{3m},\ I_{4:r:r':r'';i}^{4m}. \tag{A.2}$$

It turns out to be convenient to introduce the scalar box functions. These are defined as follows

$$I_{4:i}^{1m} = -2\frac{F_{n:i}^{1m}}{t_{i-3}^{[2]}t_{i-2}^{[2]}}, \quad I_{4:r;i}^{2m\ e} = -2\frac{F_{n:r;i}^{2m\ e}}{t_{i-1}^{[r+1]}t_i^{[r+1]} - t_i^{[r]}t_{i+r+1}^{[n-r-2]}}, \quad I_{4:r;i}^{2m\ h} = -2\frac{F_{n:r;i}^{2m\ h}}{t_{i-1}^{[2]}t_{i-1}^{[r+1]}},$$
$$I_{4:r:r';i}^{3m} = -2\frac{F_{n:r:r';i}^{3m}}{t_{i-1}^{[r+1]}t_i^{[r+r']} - t_i^{[r]}t_{i+r+r'}^{[n-r-r'-1]}}, \quad I_{4:r:r':r";i}^{4m} = -2\frac{F_{n:r:r':r";i}^{4m}}{t_i^{[r+r']}t_{i+r}^{[r'+r"]}\rho}, \tag{A.3}$$

where

$$\rho = \sqrt{1 - 2\lambda_1 - 2\lambda_2 + \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2}, \tag{A.4}$$

and

$$\lambda_1 = \frac{t_i^{[r]}t_{i+r+r'}^{[r"]}}{t_i^{[r+r']}t_{i+r}^{r'+r"}}, \quad \lambda_2 = \frac{t_{i+r}^{[r']}t_{i+r+r'+r"}^{[n-r-r'-r"]}}{t_i^{[r+r']}t_{i+r}^{r'+r"}}. \tag{A.5}$$

We have set to one a factor usually denote by r_Γ. The reason being that we are only dealing with cuts that are finite and r_Γ goes to one as ϵ, the dimensional regularization parameter, goes to 0.

A.1. Monodromy Analysis

The box functions have branch cuts and monodromies. These monodromies can be computed as the imaginary part of the box function in an appropriate kinematical region. Here we provide a general way of computing the monodromies in all channels of interest and some explicit examples.

Let us start by writing down the explicit form of the box functions,

$$\begin{aligned} F^{1m}_{n:i} =& -\frac{1}{\epsilon^2}\left[(-t^{[2]}_{i-3})^{-\epsilon}+(-t^{[2]}_{i-2})^{-\epsilon}-(-t^{[3]}_{i-3})^{-\epsilon}\right] \\ &+\mathrm{Li}_2\left(1-\frac{t^{[3]}_{i-3}}{t^{[2]}_{i-3}}\right)+\mathrm{Li}_2\left(1-\frac{t^{[3]}_{i-3}}{t^{[2]}_{i-2}}\right)+\frac{1}{2}\ln^2\left(\frac{t^{[2]}_{i-3}}{t^{[2]}_{i-2}}\right)+\frac{\pi^2}{6}, \end{aligned} \tag{A.6}$$

$$\begin{aligned} F^{2m\ e}_{n:r;i} =& -\frac{1}{\epsilon^2}\left[(-t^{[r+1]}_{i-1})^{-\epsilon}+(-t^{[r+1]}_{i})^{-\epsilon}-(-t^{[r]}_{i})^{-\epsilon}-(-t^{[r+2]}_{i-1})^{-\epsilon}\right] \\ &+\mathrm{Li}_2\left(1-\frac{t^{[r]}_{i}}{t^{[r+1]}_{i-1}}\right)+\mathrm{Li}_2\left(1-\frac{t^{[r]}_{i}}{t^{[r+1]}_{i}}\right)+\mathrm{Li}_2\left(1-\frac{t^{[r+2]}_{i-1}}{t^{[r+1]}_{i-1}}\right) \\ &+\mathrm{Li}_2\left(1-\frac{t^{[r+2]}_{i-1}}{t^{[r+1]}_{i}}\right)-\mathrm{Li}_2\left(1-\frac{t^{[r]}_{i}t^{[r+2]}_{i-1}}{t^{[r+1]}_{i-1}t^{[r+1]}_{i}}\right)+\frac{1}{2}\ln^2\left(\frac{t^{[r+1]}_{i-1}}{t^{[r+1]}_{i}}\right), \end{aligned} \tag{A.7}$$

$$\begin{aligned} F^{2m\ h}_{n:r;i} =& -\frac{1}{\epsilon^2}\left[(-t^{[2]}_{i-2})^{-\epsilon}+(-t^{[r+1]}_{i-1})^{-\epsilon}-(-t^{[r]}_{i})^{-\epsilon}-(-t^{[r+2]}_{i-2})^{-\epsilon}\right] \\ &-\frac{1}{2\epsilon^2}\frac{(-t^{[r]}_{i})^{-\epsilon}(-t^{[r+2]}_{i-2})^{-\epsilon}}{(-t^{[2]}_{i-2})^{-\epsilon}}+\frac{1}{2}\ln^2\left(\frac{t^{[2]}_{i-2}}{t^{[r+1]}_{i-1}}\right) \\ &+\mathrm{Li}_2\left(1-\frac{t^{[r]}_{i}}{t^{[r+1]}_{i-1}}\right)+\mathrm{Li}_2\left(1-\frac{t^{[r+2]}_{i-2}}{t^{[r+1]}_{i-1}}\right), \end{aligned} \tag{A.8}$$

$$\begin{aligned} F^{3m}_{n:r:r';i} =& -\frac{1}{\epsilon^2}\left[(-t^{[r+1]}_{i-1})^{-\epsilon}+(-t^{[r+r']}_{i})^{-\epsilon}-(-t^{[r]}_{i})^{-\epsilon}-(-t^{[r']}_{i+r})^{-\epsilon}-(-t^{[r+r'+1]}_{i-1})^{-\epsilon}\right] \\ &-\frac{1}{2\epsilon^2}\frac{(-t^{[r]}_{i})^{-\epsilon}(-t^{[r']}_{i+r})^{-\epsilon}}{(-t^{[r+r']}_{i})^{-\epsilon}}-\frac{1}{2\epsilon^2}\frac{(-t^{[r']}_{i+r})^{-\epsilon}(-t^{[r+r'+1]}_{i-1})^{-\epsilon}}{(-t^{[r+1]}_{i-1})^{-\epsilon}}+\frac{1}{2}\ln^2\left(\frac{t^{[r+1]}_{i-1}}{t^{[r+r']}_{i}}\right) \\ &+\mathrm{Li}_2\left(1-\frac{t^{[r]}_{i}}{t^{[r+1]}_{i-1}}\right)+\mathrm{Li}_2\left(1-\frac{t^{[r+r'+1]}_{i-1}}{t^{[r+r']}_{i}}\right)-\mathrm{Li}_2\left(1-\frac{t^{[r]}_{i}t^{[r+r'+1]}_{i-1}}{t^{[r+1]}_{i-1}t^{[r+r']}_{i}}\right), \end{aligned} \tag{A.9}$$

$$\begin{aligned} F^{4m}_{n:r:r':r";i} =& \frac{1}{2}\Bigg(-\mathrm{Li}_2(K_{-++})+\mathrm{Li}_2(K_{-+-})-\mathrm{Li}_2\left(-\frac{1}{\lambda_1}K_{---}\right)+\mathrm{Li}_2\left(-\frac{1}{\lambda_1}K_{--+}\right) \\ &-\frac{1}{2}\ln\left(\frac{\lambda_1}{\lambda_2^2}\right)\ln\left(\frac{K_{+-+}}{K_{+--}}\right)\Bigg) \end{aligned} \tag{A.10}$$

where

$$K_{\epsilon_1,\epsilon_2,\epsilon_3} = \frac{1}{2}\left(1+\epsilon_1\lambda_1+\epsilon_2\lambda_2+\epsilon_3\rho\right). \tag{A.11}$$

In these formulas, $\mathrm{Li}_2(x)$ denotes the dilogarithm function as defined by Euler, i.e., $\mathrm{Li}_2 = -\int_0^x \ln(1-z)dz/z$.

We now turn to the study of the monodromies. The discussion that follows is applicable for the first three sets of scalars box functions. The four-mass scalar box function is more complicated and the analysis of its analytic structure is done at the end of this appendix.

Suppose that we want to compute the monodromy around the branch cut in the $t_i^{[r]}$ channel. The appropriate kinematical regime is that in which $t_i^{[r]}$ is positive and all other invariants are negative. The monodromy is related to the imaginary part of the function in this special regime[15]. We can compute the imaginary part of the dilogarithms by using Euler's identity,

$$\mathrm{Li}_2(1-z) = \mathrm{Li}_2(z) + \frac{\pi^2}{6} - \ln z \ln(1-z). \tag{A.12}$$

The dilogarithm $\mathrm{Li}_2(z)$ is real for $z<1$, therefore, for any function g which is negative in the kinematical regime of interest we have

$$\begin{aligned}
\Delta\mathrm{Li}_2\left(1-gt_i^{[r]}\right) &= \mathrm{Im}|_{t_i^{[r]}>0}\mathrm{Li}_2\left(1-gt_i^{[r]}\right) = -\pi\ln\left(1-gt_i^{[r]}\right),\\
\Delta\mathrm{Li}_2\left(1-\frac{g}{t_i^{[r]}}\right) &= -\mathrm{Im}|_{t_i^{[r]}>0}\mathrm{Li}_2\left(1-\frac{g}{t_i^{[r]}}\right) = \pi\ln\left(1-\frac{g}{t_i^{[r]}}\right),\\
\Delta\ln^2\left(\frac{g}{t_i^{[r]}}\right) &= \mathrm{Im}|_{t_i^{[r]}>0}\ln^2\left(\frac{g}{t_i^{[r]}}\right) = 2\pi\ln\left(\frac{g}{t_i^{[r]}}\right).
\end{aligned} \tag{A.13}$$

These formulas cover all the monodromies of the finite part of the box functions.

The infrared divergent terms of the form

$$\frac{1}{\epsilon^2}(-t_i^{[r]})^{-\epsilon} \tag{A.14}$$

also develop an imaginary part

$$\mathrm{Im}|_{t_i^{[r]}>0}\left(\frac{1}{\epsilon^2}(-t_i^{[r]})^{-\epsilon}\right) = \frac{2\pi}{\epsilon}\ln t_i^{[r]} + \mathcal{O}(\epsilon^0). \tag{A.15}$$

[15] The sign depends on the direction the cut is crossed.

In most of the paper we ignore this contribution but in section 3.1 it is explicitly shown to cancel out.

Finally, let us consider some useful examples of the monodromies around the $t_1^{[3]}$ cut:

1. Consider the "one-mass" box function (A.6). The only such function with a cut in this channel[16] is $F^{1m}_{n:4}$,

$$\Delta F^{1m}_{n:4} = -\ln\left(1 - \frac{t_1^{[3]}}{t_1^{[2]}}\right) - \ln\left(1 - \frac{t_1^{[3]}}{t_2^{[2]}}\right). \tag{A.16}$$

2. Consider the "two-mass-easy" box function (A.7). There are three such functions with a cut in this channel: $F^{2m\ e}_{n:2;2}$,$F^{2m\ e}_{n:3;1}$, and $F^{2m\ e}_{n:2;1}$. let us study the first,

$$\Delta F^{2m\ e}_{n:2;2} = \ln\left(1 - \frac{t_2^{[2]}}{t_1^{[3]}}\right) + \ln\left(1 - \frac{t_1^{[4]}}{t_1^{[3]}}\right) - \ln\left(1 - \frac{t_2^{[2]}t_1^{[4]}}{t_1^{[3]}t_2^{[3]}}\right) + \ln\left(-\frac{t_1^{[3]}}{t_2^{[3]}}\right). \tag{A.17}$$

Even though we have introduced here a special notation for the monodromy, i.e., Δ, in the rest of the paper we make a somewhat abuse of notation and call it the imaginary part in the channel of interest.

A.2. Four-Mass Scalar Box Function

As mentioned before, the four-mass scalar box function is special. It is the only scalar box function that has square roots of the kinematical invariants in the arguments of the logarithms and dilogarithms.

Here we consider a given four-mass scalar box function and find to which cuts it can contribute and the form of the corresponding monodromy. We then turn to the particular case of the $(1,2,3)$ cut.

Let us consider $F^{4m}_{n:r:r':r'';i}$. The function depends on six kinematical invariants, i.e. $t_i^{[r]}$, $t_{i+r+r'}^{[r'']}$, $t_{i+r}^{[r']}$, $t_{i+r+r'+r''}^{[n-r-r'-r'']}$, $t_i^{[r+r']}$, and $t_{i+r}^{[r'+r'']}$.

Naively, one might think that $F^{4m}_{n:r:r':r'';i}$ has branch cuts in the six channels defined by the kinematical regime where anyone of the six invariants is positive and the rest are negative. This is not possible physically as the box function can only contribute to four different channels. As we now study in detail this is indeed the case; in the kinematical regime where $t_{i+r}^{[r']}$ or $t_{i+r+r'+r''}^{[n-r-r'-r'']}$ are positive and the rest negative, the four-mass box function does not develop any imaginary part.

[16] For $n=6$ there are two possibilities: $F^{1m}_{6:1}$ and $F^{1m}_{6:4}$. For $n=5$ there are three: $F^{1m}_{5:1}$, $F^{1m}_{5:4}$, and $F^{1m}_{5:2}$.

First note that the box function depends on the kinematical invariants only through the combinations

$$\lambda_1 = \frac{t_i^{[r]} t_{i+r+r'}^{[r"]}}{t_i^{[r+r']} t_{i+r}^{r'+r"}}, \quad \lambda_2 = \frac{t_{i+r}^{[r']} t_{i+r+r'+r"}^{[n-r-r'-r"]}}{t_i^{[r+r']} t_{i+r}^{r'+r"}}. \tag{A.18}$$

This implies that conditions on the kinematical invariants translate into conditions on λ_1 and λ_2. There are only three cases to consider:

1. The $t_i^{[r]}$-cut and the $t_{i+r+r'}^{[r'']}$-cut are characterized by $\lambda_1 < 0$ and $\lambda_2 > 0$.
2. The $t_{i+r}^{[r']}$-cut and the $t_{i+r+r'+r''}^{[n-r-r'-r'']}$-cut are characterized by $\lambda_1 > 0$ and $\lambda_2 < 0$.
3. The $t_i^{[r+r']}$-cut and the $t_{i+r}^{[r'+r'']}$-cut are characterized by imposing $\lambda_1 < 0$ and $\lambda_2 < 0$.

We have to consider the arguments of each of the dilogarithms and logarithms in (A.10). For dilogarithms we have to determine whether the arguments are larger or smaller than one. Recall that

$$\text{Im Li}_2(x) = \begin{cases} 0 & x \leq 1 \\ -\pi \ln(x) & x > 1 \ . \end{cases} \tag{A.19}$$

For the logarithm we have to determine whether the arguments are positive or negative.

All this discussion has implicitly assumed that the arguments are real. Note that this is indeed the case. It is not difficult to check that ρ is always real for any real values of λ_1 and λ_2 in the three regimes of interest.

We find that a table is the most convenient way of presenting all this information. If a given argument produces an imaginary part in a given regime we write "Yes" otherwise we write "No".

	K_{-++}	K_{-+-}	$-\frac{1}{\lambda_1}K_{---}$	$-\frac{1}{\lambda_1}K_{--+}$	λ_1/λ_2^2	K_{+-+}/K_{+--}
$\lambda_1 < 0$ $\lambda_2 > 0$	Yes	No	No	Yes	Yes	Yes
$\lambda_1 > 0$ $\lambda_2 < 0$	No	No	No	No	No	No
$\lambda_1 < 0$ $\lambda_2 < 0$	Yes	No	No	Yes	Yes	No

From this table, it is clear that $F^{4m}_{n:r:r':r'';i}$ does not have an imaginary part neither in the $t_{i+r}^{[r']}$-channel not in the $t_{i+r+r'+r''}^{[n-r-r'-r'']}$-channel.

Finally, note that for any cut of the form studied in section 2, we expect the action of the appropriate collinear operator on it to give a rational function. From our table and (A.19) it easy to see that there is no kinematical regime where a pole of the form $F + \sqrt{K}$ would not be present. This proves that the four-mass scalar box function can not have a nonzero coefficient if it participates in this class of cuts.

Appendix B. Detailed Computation Of $[F_{123}, \eta]C_{123}$

In this appendix we provide the details of the computation leading to (3.8). The computation involves all the details that could be encountered in the general discussion of section 2 leading to (2.17). In particular, the jacobian factor $\mathcal{J}$ is computed here.

As mentioned in general in section 2 and in our example in section 3, the computation does not involve any actual integration. There are enough delta functions to localize the integral completely. Therefore, the only thing to do is to exhibit all delta functions explicitly, check that their support is in the region of integration and compute all possible jacobians.

Our starting point here is (3.7),

$$\int d\mu \frac{1}{\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}\langle 2\ 3\rangle[\partial_1, \eta]\left(\frac{1}{\langle \ell_1\ 1\rangle}\right) =$$
$$\int d^4\ell_1\delta^{(+)}(\ell_1^2)\int d^4\ell_2\delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)\frac{1}{\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}\langle 2\ 3\rangle[\partial_1, \eta]\left(\frac{1}{\langle \ell_1\ 1\rangle}\right). \tag{B.1}$$

Recall that $P = p_1 + p_2 + p_3$.

The differential operator in this part of the integral only affects the pole in ℓ_1. Therefore, it is useful to write the integral over the future light-cone of ℓ_1 in terms of spinor variables. A convenient way of writing this integral[17] was given in section 6 of [11]. It uses a slightly different parametrization for null vectors, namely $\ell_1 = t\lambda_{\ell_1}\widetilde{\lambda}_{\ell_1}$. Then

$$\int d^4\ell_1\delta^{(+)}(\ell_1^2)\ (\bullet) = \int_0^\infty dt\ t\int \langle\lambda_{\ell_1}, d\lambda_{\ell_1}\rangle[\widetilde{\lambda}_{\ell_1}, d\widetilde{\lambda}_{\ell_1}](\bullet), \tag{B.2}$$

where the bullets represent generic arguments. Here λ_{ℓ_1} and $\widetilde{\lambda}_{\ell_1}$ are independent and become homogeneous coordinates for two different $\mathbb{CP}^1$'s. The integral is a complex integral performed over the contour $\widetilde{\lambda}_{\ell_1} = \overline{\lambda}_{\ell_1}$, i.e., the diagonal $\mathbb{CP}^1$. On the other and t scales in a way to make the measure invariant [11].

The different parametrization of ℓ_1, although convenient, it requires some care when used. Let us write the original parametrization as $\ell_1 = \lambda_{\ell_1}\widetilde{\lambda}_{\ell_1}$. Note that the integrand of (B.1) is invariant under simultaneously rescaling $\lambda_{\ell_1} \to \sigma\lambda_{\ell_1}$ and $\widetilde{\lambda}_{\ell_1} \to \sigma^{-1}\widetilde{\lambda}_{\ell_1}$. Therefore, any change of variables from the old to the new parametrization can be brought to the form $\lambda_{\ell_1,\mathrm{old}} = \lambda_{\ell_1,\mathrm{new}}$ and $\widetilde{\lambda}_{\ell_1,\mathrm{old}} = t\widetilde{\lambda}_{\ell_1,\mathrm{new}}$. This is the relation we use to convert from one to the other in what follows.

[17] An alternative possibility can be found in [26].

Let us write (B.1) using the new measure (B.2) and the new parametrization for ℓ_1

$$\begin{aligned}&\int_0^\infty dt\ t\int\langle\lambda_{\ell_1},d\lambda_{\ell_1}\rangle[\widetilde{\lambda}_{\ell_1},d\widetilde{\lambda}_{\ell_1}]\int d^4\ell_2\ \delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)\times\\&\frac{1}{t\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}\langle 2\ 3\rangle[\partial_1,\eta]\left(\frac{1}{\langle\ell_1\ 1\rangle}\right).\end{aligned}\tag{B.3}$$

Note that a factor of t has appeared in the denominator.

Having written the integral in a convenient form, let us turn to the holomorphic anomaly. Using (2.16) with $\lambda=\lambda_{\ell_1}$ and $\lambda'=\lambda_1$, we have

$$d\overline{\lambda}_{\ell_1}^{\dot{a}}\frac{\partial}{\partial\overline{\lambda}_{\ell_1}^{\dot{a}}}\frac{1}{\langle\lambda_{\ell_1},\lambda_1\rangle}=[d\overline{\lambda}_{\ell_1},\partial_{\ell_1}]\frac{1}{\langle\lambda_{\ell_1},\lambda_1\rangle}=2\pi\overline{\delta}(\langle\lambda_{\ell_1},\lambda_1\rangle).\tag{B.4}$$

This form is useful because the delta function can be used to carry out the integral over λ_{ℓ_1} and $\widetilde{\lambda}_{\ell_1}$ in (B.3).

However, the operator acting on the pole in (B.3) is not of the form required in (B.4). To bring the operator to that form note the following identity

$$[\partial_1,\eta]\left(\frac{1}{\langle\ell_1\ 1\rangle}\right)=-[\partial_{\ell_1},\eta]\left(\frac{1}{\langle\ell_1\ 1\rangle}\right).\tag{B.5}$$

Inserting this in (B.3) produces a factor of the form $[\widetilde{\lambda}_{\ell_1},d\widetilde{\lambda}_{\ell_1}][\partial_{\ell_1},\eta]$ which by Schouten's identity (2.11) becomes

$$[\widetilde{\lambda}_{\ell_1},d\widetilde{\lambda}_{\ell_1}][\partial_{\ell_1},\eta]=[\widetilde{\lambda}_{\ell_1},\partial_{\ell_1}][d\widetilde{\lambda}_{\ell_1},\eta]-[\widetilde{\lambda}_{\ell_1},\eta][d\widetilde{\lambda}_{\ell_1},\partial_{\ell_1}]\tag{B.6}$$

Note that the second term in (B.6) is precisely what we want in order to use (B.4). Luckily, the first term does not contribute. To see this note that in the contour of integration

$$[\overline{\lambda}_{\ell_1},\partial_{\ell_1}]\left(\frac{1}{\langle\ell_1\ 1\rangle}\right)=[\overline{\lambda}_{\ell_1},\overline{\lambda}_1]\delta(\langle\lambda_{\ell_1},\lambda_1\rangle)=[\overline{\lambda}_1,\overline{\lambda}_1]\delta(\langle\lambda_{\ell_1},\lambda_1\rangle)=0.\tag{B.7}$$

Let us collect all our partial result to write (B.3) as follows

$$\int d^4\ell_2\delta^{(+)}(\ell_2^2)\int_0^\infty dt\ \int\langle\lambda_{\ell_1},d\lambda_{\ell_1}\rangle\overline{\delta}(\langle\lambda_{\ell_1},\lambda_1\rangle)\delta^{(4)}(\ell_1+\ell_2-P)\frac{\langle 2\ 3\rangle[\overline{\lambda}_{\ell_1},\eta]}{\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}\tag{B.8}$$

Now the integral over λ_{ℓ_1} can be evaluated trivially by using the delta function to set $\lambda_{\ell_1}=\lambda_1$ and therefore $\ell_1=tp_1$. After this is done we are left with

$$\int d^4\ell_2\delta^{(+)}(\ell_2^2)\int_0^\infty dt\ \delta^{(4)}(tp_1+\ell_2-P)\frac{\langle 2\ 3\rangle[1,\eta]}{\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ 1]}.\tag{B.9}$$

The integral over ℓ_2 is trivial because we can use $\delta^{(4)}(tp_1 + \ell_2 - P)$ to get

$$\frac{\langle 2\ 3\rangle[1\ \eta]}{[n\ 1]} \int_0^\infty dt\ \delta^{(+)}(\ell_2^2)\ \frac{1}{\langle 3\ \ell_2\rangle[\ell_2\ 4]} \tag{B.10}$$

where $\ell_2 = P - tp_1$.

The remaining delta function can be written in a more convenient form

$$\delta^{(+)}(\ell_2^2) = \delta\left(t(2p_1 \cdot P) - P^2\right)\vartheta(E_{\ell_2}) \tag{B.11}$$

where E_{ℓ_2} is the energy component of ℓ_2 and $\vartheta(x)$ is 1 for $x \geq 0$ and 0 for $x < 0$.

Note that $(2p_1 \cdot P) = t_1^{[3]} - t_2^{[2]}$. Recalling that this computation is done in the kinematical regime where $t_1^{[3]} > 0$ and all other invariants are negative, in particular $t_2^{[2]} < 0$, it is clear that $(2p_1 \cdot P)$ is always positive. Therefore it can easily be pulled out of the delta function

$$\delta^{(+)}(\ell_2^2) = \frac{1}{2p_1 \cdot P}\delta\left(t - \frac{P^2}{2p_1 \cdot P}\right)\vartheta(E_{\ell_2}). \tag{B.12}$$

The integral over t in (B.10) is again trivial and nonzero, for $t = t_1^{[3]}/(t_1^{[3]} - t_2^{[2]})$ always satisfies the following condition: $0 < t < 1$. This also implies that E_{ℓ_2} is always positive. To see this recall that $\ell_2 = (1-t)p_1 + p_2 + p_3$ and that all p_1, p_2, and p_3 have positive energy.

Combining all this we find that (B.10) equals

$$\frac{\langle 2\ 3\rangle[1\ \eta]}{[n\ 1]\langle 3\ \ell_2\rangle[\ell_2\ 4](2p_1 \cdot P)}. \tag{B.13}$$

Writing $[\ell_2\ 4] = (2p_4 \cdot \ell_2)/\langle \ell_2\ 4\rangle$ we get

$$\langle 2\ 3\rangle \frac{[1\ \eta]}{[n\ 1]} \frac{\langle 4\ \ell_2\rangle}{\langle 3\ \ell_2\rangle}\ \frac{1}{(2p_1 \cdot P)(2p_4 \cdot P) - P^2(2p_1 \cdot p_4)}. \tag{B.14}$$

This formula can be further simplified by noticing the following identities

$$(2p \cdot P)(2q \cdot P) - P^2(2p \cdot q) = \langle q|P|p]\langle p|P|q] \tag{B.15}$$

and

$$\frac{\langle 4\ \ell_2\rangle}{\langle 3\ \ell_2\rangle} = \frac{\langle 4|P|1]}{\langle 3|P|1]} = \frac{\langle 4|P|1]}{\langle 3\ 2\rangle[2\ 1]}. \tag{B.16}$$

Using (B.15) with $p = p_4$ and $q = p_1$, and (B.16) in (B.14) we finally find (3.8)

$$\int d\mu \frac{1}{\langle 3\ \ell_2\rangle[\ell_2\ 4][n\ \ell_1]}\langle 2\ 3\rangle[\partial_1, \eta]\left(\frac{1}{\langle \ell_1\ 1\rangle}\right) = \frac{[1\ \eta]}{\langle 1|P|4][n\ 1][1\ 2]}. \tag{B.17}$$

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